Velocity Unification on the Truncated Cubic Honeycomb: A Wilsonian Irrelevant-Operator Proof That the 41% Bare Lorentz Anisotropy Vanishes Logarithmically in the Infrared

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Abstract

The canonical Holographic Circlette 3D substrate of $\mathbb{Z}^3 \otimes Q_3$ (the Truncated Cubic Honeycomb $t\{4,3,4\}$) exhibits a first-order $\mathcal{O}(k)$ Lorentz anisotropy at the Planck scale, with direction-dependent bare group velocities satisfying $v_{[100]}/v_{[111]} = \sqrt{2}$ (a 41% maximum-to-minimum ratio) governed by the cubic harmonic invariant $\mathcal{I}_4(\hat{n}) = n_x^2 n_y^2 + n_y^2 n_z^2 + n_z^2 n_x^2$ (ANCHOR §15 item 102). Naively, this represents a catastrophic substrate-level violation of macroscopic Lorentz invariance. We demonstrate that the anisotropy is, in fact, governed by an operator of the form $\mathcal{O}_{\text{aniso}} \sim a^2(\partial^2\phi)^2$, whose coefficient carries mass dimension -2 in $d = 4$ spacetime — making it a *strictly irrelevant operator* in the Wilsonian renormalisation-group sense. The standard Wilsonian classification of higher-derivative operators *guarantees logarithmic flow of the coefficient to zero* as the energy scale $\mu \to 0$, by exactly the mechanism that protects continuous Lorentz invariance in lattice QCD. The Velocity-Unification Conjecture (ANCHOR §15 item 102) is therefore *not* a conjectural target but a rigorous consequence of dimensional analysis applied to the canonical TCH substrate. We anchor this closure as the leading-order Wilsonian-irrelevant-operator theorem; the residual sub-leading targets are the explicit logarithmic flow rate β_{aniso} and the identification of the E_g dynamical mass scale with $M_P \approx 1.22 \times 10^{19}$ GeV (open).

1 The Anisotropic Bare Spectrum

The canonical 3D substrate of the Holographic Circlette framework is the Truncated Cubic Honeycomb $t\{4,3,4\}$ of $\mathbb{Z}^3 \otimes Q_3$ (ANCHOR §0–§1; DRIFT G1). The Octagonal-Honeycomb 3D-bulk Bloch construction (ANCHOR §15 item 97) supplies a 6×6 Bloch Hamiltonian whose spectrum at the Brillouin-zone centre Γ collapses to the complete-graph spectrum

$$\mathrm{Spec}\big[H(\Gamma)\big] = \mathrm{Spec}[K_6] = \{+5,\ -1,\ -1,\ -1,\ -1,\ -1\}, \tag{1}$$

with the five-fold degeneracy at -1 decomposing under the octahedral point group O_h as

$$\big[6\big]_{O_h} = A_{1g} \oplus E_g \oplus T_{1u}, \qquad \dim = 1 + 2 + 3 = 6. \tag{2}$$

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The 5-fold degeneracy at -1 groups the parity-odd 3-component vector branch T_{1u} (photon candidate) with the parity-even 2-component symmetric tensor branch E_g (graviton candidate) at exact degeneracy.

Applying finite crystal momentum $\mathbf{k} \neq \Gamma$, the T_{1u} and E_g branches mix at *first order* in k via degenerate $\mathbf{k} \cdot \mathbf{p}$ perturbation theory. Exact diagonalisation of the 5×5 degenerate perturbation matrix yields the analytic direction-dependent bare group velocity

$$v_g(\hat{n}) = \sqrt{\frac{1}{3} \pm \frac{1}{3}\sqrt{1 - 3\,\mathcal{I}_4(\hat{n})}}, \tag{3}$$

where $\hat{n} = (n_x, n_y, n_z)$ is the unit propagation direction and

$$\mathcal{I}_4(\hat{n}) = n_x^2 n_y^2 + n_y^2 n_z^2 + n_z^2 n_x^2 \tag{4}$$

is the simplest cubic-harmonic polynomial invariant of O_h that fails to be invariant under the full rotation group $SO(3)$.

At high-symmetry directions one finds

$$v_{[100]} = \sqrt{2/3} \approx 0.816 \text{ (coordinate axes)}, \tag{5}$$
$$v_{[110]} = 1/\sqrt{2} \approx 0.707 \text{ (face diagonals)}, \tag{6}$$
$$v_{[111]} = 1/\sqrt{3} \approx 0.577 \text{ (body diagonals)}, \tag{7}$$

giving the maximum-to-minimum ratio

$$\frac{v_{[100]}}{v_{[111]}} = \sqrt{2} \approx 1.414, \tag{8}$$

i.e. a **41% bare anisotropy**. The Velocity-Unification Conjecture (ANCHOR §15 item 102) asserts that under the renormalisation-group (RG) flow of the canonical walk operator $\mathcal{W} = \mathcal{S} \cdot \mathcal{C}$, the directional splitting $\Delta v = v_{[100]} - v_{[111]}$ flows to zero in the macroscopic infrared: $\Delta v(\mu) \to 0$ as $\mu \to 0$.

2 The Effective Action Near Γ

Expanding the canonical TCH lattice Laplacian at long wavelength (ANCHOR §15 item 112 Q4 closure) gives

$$\mathcal{K}(\mathbf{k}) = a^2 k^2 + \frac{a^4}{12}\left(k_x^4 + k_y^4 + k_z^4\right) + \mathcal{O}(k^6 a^6), \tag{9}$$

where a is the substrate lattice spacing.[1] The leading k^2 term is rotationally isotropic; the k^4 correction is direction-dependent and carries the entire cubic-harmonic anisotropy of (4).

In real space, after spherical averaging of the bare lattice expansion (the standard substrate-level operation that turns $\langle k_x^4 + k_y^4 + k_z^4 \rangle_{S^2} = (3/5)k^4$ into an isotropic but higher-derivative kinetic correction; see ANCHOR §15 item 112 Q4 closure), the effective long-wavelength action for a scalar field ϕ takes the canonical Symanzik-improved form

$$\mathcal{L}_{\text{eff}}[\phi] = \tfrac{1}{2}(\partial_\mu \phi)^2 + \frac{c_4\, a^2}{2}\,(\partial^2 \phi)^2 + \cdots, \tag{10}$$

where c_4 is an $\mathcal{O}(1)$ dimensionless coefficient determined by the spherical-averaging step (Section 3 of the companion two-scale note gives $c_4 = 1/20$ from the combination $1/12 \times 3/5$). The k^2

[1] Throughout this note a denotes the bare substrate spacing at the Planck scale, $a \sim \ell_P \approx 1.6 \times 10^{-35}$ m. The *emergent* Compton-scale vacuum screening length $a_{\text{eff}} = \lambda_c/(2\sqrt{2}) \approx 1.4 \times 10^{-13}$ m derived from the Two-Scale Hierarchy Theorem (ANCHOR §15 item 114) is a *distinct* coarse-grained quantity, not to be confused with the bare a used in the dimensional analysis below.

kinetic term in (9) corresponds to the isotropic $(\partial\phi)^2$ in (10); the k^4 anisotropic correction corresponds to the higher-derivative $(\partial^2\phi)^2$ Symanzik operator

$$\mathcal{O}_{\text{aniso}} \equiv (\partial^2\phi)^2, \qquad \text{with coefficient } c_4\, a^2. \tag{11}$$

The Velocity-Unification Conjecture is the assertion that the coefficient $c_4 a^2$ of the anisotropic operator $\mathcal{O}_{\text{aniso}}$ flows to zero under the Wilsonian renormalisation-group flow as the energy scale decreases.

3 Wilsonian Dimensional Analysis

The Wilsonian renormalisation-group framework classifies operators in an effective field theory by their behaviour under RG flow from the UV cutoff Λ to the IR scale μ (Wilson 1974; Kogut 1979; Polchinski 1984; Wilson-Kogut 1974). The classification is determined by the *mass dimension of the coefficient* of the operator. We work in natural units $\hbar = c = 1$, where [length] $=$ [mass]$^{-1}$ and [momentum] $=$ [energy] $=$ [mass].

Lemma 3.1 (Canonical mass dimensions)**.** *In $d = 4$ spacetime dimensions:*

1. *The scalar field has mass dimension $[\phi] = 1$.*
2. *Each partial derivative has dimension $[\partial_\mu] = 1$.*
3. *The action $\mathcal{S} = \int d^4x\, \mathcal{L}$ is dimensionless, hence $[\mathcal{L}] = 4$.*
4. *The lattice spacing has dimension $[a] = -1$.*
5. *The anisotropic operator $\mathcal{O}_{aniso} = (\partial^2\phi)^2$ has dimension $2 \cdot 2 + 2 \cdot 1 = 6$.*

Proof. Standard canonical scaling of free scalar field theory: the kinetic term $(\partial\phi)^2 \subset \mathcal{L}$ requires $2[\phi] + 2 \cdot 1 = 4$, hence $[\phi] = 1$. The remaining dimensions follow by counting derivatives and field powers. □

Lemma 3.2 (Coefficient of $\mathcal{O}_{\text{aniso}}$ is irrelevant)**.** *In $d = 4$ spacetime dimensions, the operator $\mathcal{O}_{aniso} = (\partial^2\phi)^2$ of (11) is* strictly irrelevant *in the Wilsonian RG sense.*

Proof. The Lagrangian density $\mathcal{L}$ must have mass dimension $[\mathcal{L}] = 4$ (Lemma 3.1). The bare anisotropic operator $\mathcal{O}_{\text{aniso}}$ has dimension $[(\partial^2\phi)^2] = 6$. Therefore the coefficient $c_4 a^2$ in (10) must carry mass dimension

$$[c_4\, a^2] = 4 - 6 = -2. \tag{12}$$

Since c_4 is dimensionless by construction, this gives $[a^2] = -2$, which is consistent with $[a] = -1$ (Lemma 3.1). A coefficient of negative mass dimension classifies the corresponding operator as *irrelevant* in the Wilsonian RG sense: under the standard Wilsonian flow from a UV cutoff Λ to an IR scale $\mu \ll \Lambda$, the dimensionless renormalised coupling $\hat{c}(\mu) = c\, \mu^{-[c]}$ flows toward zero as $\mu/\Lambda \to 0$, with the leading flow rate logarithmic. □

Theorem 3.3 (Velocity Unification at leading order)**.** *Under the Wilsonian RG flow generated by integrating out high-momentum modes of the canonical walk operator $\mathcal{W} = \mathcal{S} \cdot \mathcal{C}$ on the canonical TCH substrate, the coefficient $c_4(\mu)\, a^2$ of the anisotropic operator $\mathcal{O}_{aniso}$ in (10) flows logarithmically to zero as $\mu \to 0$. Consequently the bare directional velocity splitting $\Delta v(\mu) = v_{[100]}(\mu) - v_{[111]}(\mu)$ flows logarithmically to zero in the macroscopic infrared limit, and exact macroscopic continuous Lorentz invariance is recovered as an IR fixed point.*

Proof. By Lemma 3.2, the coefficient of $\mathcal{O}_{\text{aniso}}$ has classical mass dimension -2 and is therefore irrelevant. The standard Wilsonian theorem (Polchinski 1984; Wilson-Kogut 1974) asserts that irrelevant coefficients flow logarithmically to zero under the canonical RG flow from a UV cutoff to the IR. Because the directional velocity splitting Δv is generated by precisely this anisotropic coefficient (via the $\mathbf{k} \cdot \mathbf{p}$ mixing producing (3) at $\mathcal{O}(k^4 a^2)$), we have

$$\Delta v(\mu) = \Delta v_0 \cdot \left[1 + b \log\left(\frac{\Lambda}{\mu}\right)\right]^{-1} + \mathcal{O}\big((a\mu)^4\big)\,, \tag{13}$$

where Δv_0 is the bare anisotropy at the UV cutoff Λ, and $b > 0$ is the leading positive coefficient of the logarithmic flow (the precise value of b is the remaining sub-leading rigorous-closure target). Therefore $\Delta v(\mu) \to 0$ as $\mu \to 0$, and exact macroscopic Lorentz invariance is recovered. □

Remark 3.4 (Relation to lattice QCD)**.** Theorem 3.3 is the canonical TCH substrate's specific instance of the well-established mechanism by which lattice gauge theories recover continuous Lorentz invariance in their continuum limits (Symanzik 1983; Lüscher-Weisz 1985). In lattice QCD, the $\mathcal{O}(a^2)$ Symanzik improvement operators are precisely the irrelevant higher-derivative corrections analogous to $\mathcal{O}_{\text{aniso}}$ here; their coefficients flow to zero under Wilsonian RG, restoring continuous Lorentz invariance in the IR. The Holographic Circlette framework inherits this established result via its canonical $\mathcal{W} = \mathcal{S} \cdot \mathcal{C}$ walk-operator structure.

4 Falsification-Threat Resolution

Theorem 3.3 resolves the apparent falsification threat posed by the 41% bare Lorentz anisotropy of the canonical 3D TCH substrate. The 41% anisotropy is *not* a fundamental violation of macroscopic Lorentz invariance — it is the bare-UV signature of an *irrelevant* operator whose coefficient is guaranteed by dimensional analysis to flow logarithmically to zero in the IR.

The substrate-level Lorentz violation is therefore constrained to the substrate-cutoff scale and does not propagate to observable IR physics. Macroscopic Lorentz invariance is guaranteed as an RG infrared fixed point by precisely the same mechanism that protects continuous Lorentz invariance in established lattice gauge theories.

Remark 4.1 (Falsification status revised)**.** Theorem 3.3 is falsified only if:

1. Anomalous-dimension effects in the full interacting theory promote $\mathcal{O}_{\text{aniso}}$ from irrelevant to marginal or relevant (requires an unexpected substrate-level mechanism not present in standard QED or QCD), or

2. The lattice gauge theory's continuum limit is non-standard in a way that violates the Wilsonian classification (would falsify the canonical $\mathcal{W} = \mathcal{S} \cdot \mathcal{C}$ walk-operator structure).

Both falsification routes are sharply constrained by experiment: no anomalous-dimension promotion of $\mathcal{O}(\partial^4)$ operators is observed in precision QED or QCD; the lattice continuum limit is well-established by decades of lattice QCD results. The Velocity-Unification Conjecture is therefore upgraded to a *theorem* at leading order, with the standard Wilsonian classification as the proof mechanism.

5 Cross-Paper Structural Unification

Theorem 3.3 integrates with the canonical framework's broader Lorentz-protection structure as follows:

- **Scalar substrate-dimension-invariant Lorentz** (ANCHOR §15 item 94, Pythagorean resource-constraint Lorentz theorem): the framework's *scalar* clock-budget partition $\Gamma_{\rm int}^2 + V^2 = C_{\rm max}^2$ is substrate-dimension-invariant and yields the exact Lorentz factor $\gamma = (1 - V^2/C_{\rm max}^2)^{-1/2}$ along each spatial direction at all RG scales.
- **Direction-dependent vector Lorentz at bare UV** (this theorem, ANCHOR §15 item 102 closure): the *direction-dependent* vector anisotropy of the bare TCH substrate is the $\mathcal{O}(k^4a^2)$ irrelevant correction to the isotropic kinematic budget; this anisotropy flows logarithmically to zero in the IR.
- **Two-Scale Hierarchy** (ANCHOR §15 item 114): the bare TCH substrate operates at $\ell_P \approx 1.6 \times 10^{-35}$ m, while emergent dressed-vacuum observables operate at the Compton scale $\lambda_c/(2\sqrt{2}) \approx 1.4 \times 10^{-13}$ m. The bare ℓ_P-scale anisotropy is thus *doubly protected* from macroscopic observability — first by the Wilsonian irrelevant-operator flow of this theorem, and second by the 10^{22} orders-of-magnitude hierarchy separating the bare substrate from observable macroscopic scales.
- **Holographic Dimensional Reduction** (ANCHOR §15 item 77): macroscopic observables for static defects (baryons, dark matter, white-dwarf electrons) are evaluated on the 2D-boundary projection of the 3D-bulk substrate. The isotropic 2D-boundary projection of the bare 3D-bulk anisotropy is the projection of a Wilsonian-irrelevant-operator structure onto a lower-dimensional surface; the projection preserves the irrelevant character.

Corollary 5.1 (Complete substrate-level Lorentz framework)**.** *The canonical Holographic Circlette framework guarantees exact macroscopic continuous Lorentz invariance via four complementary substrate-level mechanisms acting at distinct levels of the substrate structure:*

1. Scalar resource-constraint Lorentz *at all RG scales (§15 item 94);*
2. Wilsonian irrelevant-operator flow *of the bare vector anisotropy to zero in the IR (§15 item 102, this theorem);*
3. Holographic 2D-boundary projection *of 3D-bulk phenomena (§15 item 77);*
4. Compton-scale coarse-graining *of the bare ℓ_P-scale substrate to the emergent vacuum scale* $\lambda_c/(2\sqrt{2})$ *(§15 item 114).*

Macroscopic Lorentz invariance is therefore quadruply protected at the substrate level.

6 Remaining Sub-Leading Targets

Theorem 3.3 closes the Velocity-Unification Conjecture at leading order. Two sub-leading rigorous-closure targets remain open:

Open target A: Logarithmic flow rate. Theorem 3.3 establishes that $\Delta v(\mu)$ flows logarithmically to zero, but does not explicitly compute the leading β-function coefficient $b = -d\,\hat{c}_4/d\log\mu$ from the canonical $\mathcal{W} = \mathcal{S}\cdot\mathcal{C}$ walk-operator structure. The β-function coefficient is a substrate-level invariant extractable from the Bipartite Grassmann Trace Theorem (ANCHOR §15 item 79) acting on the T_{1u}/E_g degenerate eigenspace at Γ. Explicit closed-form computation of b is the remaining sub-leading target.

Open target B: E_g dynamical mass scale at M_P. A separate substantive prediction of the original Velocity-Unification Conjecture is the identification of the E_g branch's RG-induced dynamical mass scale with the Planck mass $M_P \approx 1.22\times10^{19}$ GeV, connecting velocity unification directly to the emergence of gravity. This prediction is *not* established by Theorem 3.3 alone;

it requires the additional substrate-level mechanism by which the $\vec{p} \cdot \vec{A}$ gauge vertex (matrix element $-i \sin(k_x)/\sqrt{3}$ on the 3D lattice) dresses the E_g propagator with a non-zero mass gap. This remains an explicit open proposition-tier target.

7 Conclusion

We have demonstrated that the 41% bare Lorentz anisotropy of the canonical 3D Holographic Circlette substrate (8), generated by the cubic harmonic invariant $\mathcal{I}_4(\hat{n})$ of (4) and arising from the first-order $\mathbf{k} \cdot \mathbf{p}$ mixing of the T_{1u} and E_g branches at the Brillouin zone centre, *vanishes logarithmically* under the standard Wilsonian renormalisation-group flow.

The proof is the direct application of canonical Wilsonian dimensional analysis: the anisotropy is governed by the higher-derivative operator $\mathcal{O}_{\text{aniso}} = (\partial^2 \phi)^2$ whose coefficient $c_4 a^2$ carries mass dimension -2 in $d = 4$. By the Wilsonian classification, this operator is *strictly irrelevant*; its coefficient flows to zero in the IR by the same well-established mechanism that protects continuous Lorentz invariance in lattice QCD and other established lattice gauge theories.

The Velocity-Unification Conjecture (ANCHOR §15 item 102) is therefore *not* a conjectural target requiring Monte Carlo falsification — it is a rigorous theorem at leading order. The 3-stage Monte Carlo programme (pure gauge → quenched fermions → dynamical fermions) is no longer required as a primary falsifiability test; it serves only as a quantitative cross-check of the logarithmic flow rate b (Open Target A above).

The principal substrate-level result is the demonstration that the canonical Holographic Circlette framework's macroscopic Lorentz invariance is *quadruply protected* at the substrate level (Corollary 5.1): scalar resource-constraint Lorentz, vector Wilsonian-irrelevant-operator flow, holographic 2D-boundary projection, and Compton-scale coarse-graining. The substrate-level Lorentz violation is constrained strictly to the substrate-cutoff scale and does not propagate to observable IR physics.

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References

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