Photons on the Emergent Truncated Cubic Honeycomb: Spontaneous Crystallisation, Line-Graph Topology, and Topological Protection of Vacuum QED

David G. Elliman[1]

[1]Neuro-Symbolic Ltd, Gloucestershire, United Kingdom , neusym.ai

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Abstract

We derive the photon physics that emerges from the Truncated Cubic Honeycomb (TCH) substrate. Rather than being added by hand, the TCH arises spontaneously from unstructured degree-3 qubit networks via simulated annealing, forming Q_3 octahedra connected by gauge bridges. At the microscopic scale, massless photons are encoded natively as gauge fluctuations on the *line graph* of this emergent lattice (where vertices equal entanglement bonds). A fixed chiral Peierls phase $\pi/4$ on every directed edge breaks time-reversal symmetry and produces non-trivial multi-band topology, with the lower seven-band complex carrying an exact integer Chern number $C = -1$ (computed via the non-Abelian Fukui-Hatsugai-Suzuki method). However, trivial global holonomies ($8 \times \pi/4 = 2\pi$) on the dual Simple Cubic (SC) gauge web protect the infrared (IR) regime from observable vacuum birefringence, evading Carroll-Field-Jackiw bounds. The microscopic theory operates as a compact $U(1)$ lattice gauge theory coupled to the $[8, 4, 4]$ matter code via minimal interaction. In the long-wavelength limit, we show it reduces exactly to the standard continuous QED Lagrangian. Numerical joint matter-photon evolution on a minimal Q_3 + bridge cluster confirms gauge-invariant propagation and dynamic back-reaction. Observable framework signatures must therefore come from matter-coupled processes, while vacuum entanglement reproduces standard quantum mechanics perfectly.

Contents

1 Introduction

The neuro-symbolic framework proposes that physical reality is built on an emergent topological substrate [1]. Recent work has accumulated quantitative successes in the matter sector [2], with predictions for fermion mass ratios, CKM and PMNS matrix elements, and the phase $\sin(2\beta)$ matching experiment without adjustable continuous parameters. The present paper formalises the framework's photon (gauge) sector.

A historical challenge in discrete spacetime models is that photons are often mathematically cumbersome, requiring the *ad hoc* addition of auxiliary continuous vector fields layered over a rigid grid. Here, we demonstrate that photons are not inserted by hand; they emerge naturally as fluctuations on the exact same entanglement bonds that drive the spontaneous crystallisation of the substrate.

Our analysis proceeds by establishing the spontaneous microscopic emergence of the Truncated Cubic Honeycomb (TCH) (Section 2), then deriving the photonic degrees of freedom on the line graph of the substrate and exploring its multi-band topology, including a non-trivial Chern number on the lower band complex (Section 3). Transitioning to the macroscopic effective field theory, we demonstrate that the long-wavelength limit on the dual Simple Cubic (SC) gauge web yields standard massless QED dispersion and Coulomb potentials (Section 4). We then demonstrate minimal gauge-matter coupling and exact algebraic reduction to the continuum QED Lagrangian, supported by numerical joint evolution (Section 5). Finally, we prove a strict topological protection theorem: despite the presence of local chiral phases, specific macroscopic holonomies forbid parity-violating effective operators in the vacuum, satisfying cosmological bounds unconditionally (Section 6).

2 Spontaneous Emergence of the TCH Substrate

The Truncated Cubic Honeycomb (TCH) is a uniform 3D Archimedean honeycomb ($t\{4, 3, 4\}$) alternating between regular octahedra (matter cells) and truncated cubes (gauge cells).

Crucially, the TCH lattice is not an axiomatic background assumption; it emerges dynamically from the framework's unstructured graph-theoretic foundations. Starting from a 3-regular network of entanglement bonds (qubits), spatial geometry emerges via simulated annealing. The structural Hamiltonian driving this crystallisation assigns energy penalties to specific local loop closures:

$$E(G) = -w_4 \sum(\text{4-cycles}) - w_6 \sum(\text{6-cycles}) + \lambda \sum_v (\deg(v) - 3)^2 \quad (1)$$

Minimising this energy drives the network to self-assemble into $\lfloor N/8 \rfloor$ copies of the Q_3 hypercube (the face-adjacency graph of the regular octahedron) [3].

When inter-cluster gauge bridges are permitted to form, these Q_3 octahedra spontaneously tile Euclidean space into the full 3D orthogonal-octagon honeycomb (the TCH), with exact

Wyckoff positions governed by Coxeter/Conway geometry [5]. At each vertex of the tiling, exactly five cells meet: one octahedron and four truncated cubes.

3 Microscopic Topology: Photons on the Line Graph

3.1 Photonic Degrees of Freedom

The natural domain for photonic degrees of freedom is the **line graph** $L(\mathrm{TCH})$ of the emergent substrate. In this construction, the vertices of the line graph correspond to the original TCH edges (the entanglement bonds defining the crystallisation), and the edges of the line graph correspond to pairs of TCH edges sharing a vertex.

Each Q_3 octahedron natively contributes 12 photonic modes; the gauge bridges connecting the octahedra contribute additional routing links. The tight-binding photonic Hamiltonian on this edge graph is:

$$H_{\mathrm{tb}}(\mathbf{k}) = \sum_{\langle v,w\rangle} J\Big(a_v^\dagger a_w\, e^{i\phi_{vw}} + \mathrm{h.c.}\Big) + \sum_{\mathbf{R}} H_{\mathrm{bridge}}(\mathbf{R}) e^{i\mathbf{k}\cdot\mathbf{R}} \tag{2}$$

Here, $\phi_{vw} = \pm\pi/4$ is the framework's fundamental chiral phase. It is entirely determined by the local turn rule at the shared vertex, enforcing a right-hand rule strictly consistent with the C_{4v} reduction on the underlying 4.8.8 tiling.

Listing 1: Line-graph construction with chiral phases

```
# From the test suite (TestLatticeBlochStructure)
H_primal = H_int() + H_ext(kx, ky)   # 8x8 Bloch Hamiltonian
# Line-graph construction with chiral phases applied to each directed hop
H_ph = build_line_graph_with_phases(H_primal, chiral_phase=np.pi/4)
```

3.2 Chiral Phases and Local Topology

On a single Q_3 line graph, the chiral $\pi/4$ phases break time-reversal symmetry: explicit verification confirms that $H(-\mathbf{k}) \neq H^*(\mathbf{k})$ once the phases are applied. The unphased line graph at the Brillouin-zone centre Γ has the cuboctahedral graph spectrum, with twelve eigenvalues:

$$\text{Unphased: } [-2,-2,-2,-2,-2,\ 0,0,0,\ 2,2,2,\ 4] \tag{3}$$

Applying the directed chiral $\pi/4$ phases breaks TR symmetry and partially lifts the degeneracies, separating the four singleton modes while preserving three T_d-protected three-fold multiplets:

$$\text{Phased: } [-3.864,\ -2,-2,-2,\ 0,0,0,\ 1.035,\ 2,2,2,\ 2.828] \tag{4}$$

Away from Γ, the singleton modes disperse and the multiplets remain locally three-fold degenerate, connected to the singletons by Dirac-like band touchings [15] at the C_{4v}-symmetric points $(k_x, k_y) = (\pm\pi/2, 0)$ and $(0, \pm\pi/2)$. Numerical verification confirms that the gap $E_1(\mathbf{k}) - E_0(\mathbf{k})$ closes linearly with $|\mathbf{k} - \mathbf{k}_*|$ at each touching point, consistent with linear (massless) dispersion.

The presence of multi-band touchings at the C_{4v} points means that individual bands are not topologically isolated, and per-band Chern numbers in the standard sense are not well-defined. The robust topological invariant is the multi-band Chern number, computed via the non-Abelian Fukui–Hatsugai–Suzuki method [7] on the projector onto an isolated band complex:

$$C_{\mathcal{S}} = \frac{1}{2\pi} \sum_{\text{plaquettes}} \arg\det\big(M_{01}^{(\mathcal{S})}\, M_{12}^{(\mathcal{S})}\, M_{23}^{(\mathcal{S})}\, M_{30}^{(\mathcal{S})}\big), \tag{5}$$

where $M_{ij}^{(\mathcal{S})} = U_{\mathcal{S}}(\mathbf{k}_i)^\dagger U_{\mathcal{S}}(\mathbf{k}_j)$ are the overlap matrices of the band-complex projectors at adjacent BZ plaquette corners. Applied to the lower seven-band manifold $\mathcal{S}_7 = \{0, 1, \ldots, 6\}$ (the singlet at -3.864 together with the two three-fold multiplets at -2 and 0, separated from the upper five bands by a clean gap), this yields an exact integer Chern number

$$C_{\mathcal{S}_7} = -1, \tag{6}$$

confirming that the framework exhibits non-trivial multi-band topology at the microscopic substrate scale [14]. Total Chern across all twelve bands sums to zero, as required by closure of the Hilbert space.

4 Macroscopic Coarse-Graining: The SC Gauge Web

4.1 Dual Reduction to Simple Cubic Lattice

While the microscopic dynamics reside on the line graph, the macroscopic propagation of the gauge field coarse-grains over the dual of the truncated cubes. Each truncated cube has 6 octagonal faces shared with its neighbours. Connecting the centres of adjacent truncated cubes forms an exact Simple Cubic (SC) Bravais lattice.

Proposition 1 (Gauge web $\equiv$ SC lattice)**.** *For macroscopic $U(1)$ gauge fluxes, the dual graph of the gauge sector of the TCH reduces precisely to the Simple Cubic lattice.*

The pure-gauge action on this SC web is the standard Wilson action [8]. For Gaussian fluctuations, this yields the discrete SC Laplacian structure factor in momentum space:

$$\mathcal{K}(\mathbf{k}) = 6 - 2\left[\cos(k_x a) + \cos(k_y a) + \cos(k_z a)\right] \tag{7}$$

where a is the SC lattice spacing.

4.2 Photon Dispersion and the Coulomb Potential

The free-photon equation of motion gives the dispersion $\omega(\mathbf{k}) = (c/a)\sqrt{\mathcal{K}(\mathbf{k})}$. For wavelengths much larger than the lattice spacing ($|\mathbf{k}|a \ll 1$), Taylor expansion perfectly recovers standard continuum physics:

$$\omega(\mathbf{k}) \approx c|\mathbf{k}| \quad \text{(massless, isotropic, linear)}. \tag{8}$$

The first structural correction is $O(k^2a^2)$. Assuming a is on the order of the Planck length, this correction is $\sim 10^{-56}$ for visible light, ensuring exact Lorentz invariance is restored to all achievable precision.

Furthermore, the inverse Fourier transform of the static Laplacian yields the Coulomb propagator $G(\mathbf{r})$, strictly recovering the exact $1/(4\pi r)$ potential at long distances, governed at the origin by the Watson integral $G(0) \approx 0.2527$ [10].

4.3 Polarisation Structure from Substrate Symmetry

The SC gauge web supports two transverse modes from gauge invariance. Each link of the SC gauge web corresponds to one octagonal face of TCH, which has C_8 rotational symmetry around its centre. The relevant subgroup for a vector field is $C_4 \subset C_8$.

The two non-trivial E-representations E_+ and E_- correspond to states that pick up phases $+i$ and $-i$ under the C_4 rotation, identically mapping to right- and left-circular polarisation states:

$$|R\rangle \equiv |E_+\rangle, \quad |L\rangle \equiv |E_-\rangle \tag{9}$$

For a parametric down-conversion source, the framework predicts generation of Bell states in this basis, which mathematically reproduces the standard CHSH Tsirelson bound $S = 2\sqrt{2}$ [16].

5 Minimal Gauge-Matter Coupling and the Exact QED Limit

5.1 Gauge-Matter Interaction

Matter fields ψ (encoded via the $[8,4,4]$ error-correcting code) occupy the vertices and faces of the Q_3 octahedra. Coupling to the photonic line graph is achieved via standard minimal gauge-matter interaction. The gauged walk operators take the form:

$$T_x^{\text{gauged}} = T_x \otimes U_l, \quad T_y^{\text{gauged}} = T_y \otimes U_l, \quad T_z^{\text{gauged}} = T_z \otimes U_l \tag{10}$$

where $U_l = e^{i\theta_l}$ is the compact $U(1)$ link operator on edge l.

The complete interacting system is governed by the standard Kogut–Susskind Hamiltonian [9]:

$$H_{\text{ph}} = \frac{g^2}{2}\sum_l E_l^2 + \frac{1}{g^2}\sum_p \Big(1 - \text{Re}\prod_{l\in\partial p} U_l\, e^{i\Phi_p}\Big) \tag{11}$$

5.2 Algebraic Reduction to the Continuum

To demonstrate that the framework organically produces standard electrodynamics, we perform a long-wavelength expansion around the vacuum ($\theta_l \approx 0$), projecting ψ to low-energy codewords.

Evaluating the effective 2×2 Hamiltonian in the $\{A_1 \text{ matter}, E \text{ gauge}\}$ subspace yields:

$$\langle A_1|H_{\text{ext}}(k_x,0)|E_x\rangle \approx -\frac{i}{2}k_x \tag{12}$$

This directly corresponds to the canonical minimal-coupling vertex ip_xA_x. Consequently, in the continuum limit, the discrete lattice dynamics mathematically reduce exactly to the standard QED Lagrangian:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\big(i\gamma^\mu D_\mu - m\big)\psi \tag{13}$$

with the exact covariant derivative $D_\mu = \partial_\mu - ieA_\mu$.

5.3 Numerical Confirmation

These analytic derivations are robustly verified via exact numerical joint evolution in the framework's test suite [4].

Listing 2: Joint matter-photon evolution

```
# run_joint_evolution.py (excerpt)
H_matter_gauged = T_x_gauged + T_y_gauged + T_z_gauged   # tensor U_l
H_ph = photonic_tight_binding_line_graph()
H_total = H_matter_gauged + H_ph

# Time evolution with expm_multiply shows gauge back-reaction
psi_t = expm_multiply(-1j * H_total * t, psi_0)
```

Time evolution of a minimal Q_3 + bridge cluster (comprising 25 photonic modes and a 96-dimensional matter subspace) explicitly demonstrates dynamic gauge back-reaction. Photon occupation on the inter-cluster bridge actively modulates the spatial spreading of the matter wave: the probability amplitude of matter remaining on the origin Q_3 cluster rises significantly from 17% to 37% when gauge coupling is activated, confirming strict gauge-invariant propagation.

6 Topological Protection of Vacuum Physics

6.1 The CFJ Operator and Trivial Holonomy

Because the line-graph substrate carries a local chiral phase of $\pi/4$ producing a non-trivial multi-band Chern number $C_{\mathcal{S}_7} = -1$, one might naively expect massive parity violation in vacuum photon propagation. In standard Effective Field Theory (EFT), this chiral background generates a Carroll-Field-Jackiw (CFJ) operator [11]:

$$\mathcal{L}_{\text{CFJ}} = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu A_\rho \, k_\sigma \tag{14}$$

Cosmological birefringence bounds constrain $|k_\sigma| \lesssim 10^{-43}$ GeV [12,13]. A crude coarse-graining of a Planck-scale $\pi/4$ phase would predict $k_\sigma \sim 10^{18}$ GeV, which would fatally falsify the framework.

The framework evades this catastrophe through strict geometric cancellation. We establish the **Trivial Holonomy Theorem**:

Proposition 2 (Trivial Holonomy)**.** *The framework's chiral phase $\pi/4$ produces trivial holonomies (modulo 2π) at every macroscopic geometric level of the SC gauge web.*

Proof. On the TCH gauge web, natural closed loops evaluate as follows:

- **Octagonal faces:** 8 edges $\times\, \pi/4 = 2\pi \equiv 0 \pmod{2\pi}$.
- **SC plaquettes** (enclosing a Q_3 octahedron): 8 faces $\times\, 3\pi/4 = 6\pi \equiv 0 \pmod{2\pi}$.

All closed loops natively constructible by macroscopic photons evaluate to $0 \pmod{2\pi}$. □

Theorem 1 (Topological Protection of the Infrared)**.** *Because all closed macroscopic loops on the TCH SC web sum to trivial phases modulo 2π, the effective CFJ background vector $k_\sigma \to 0$. The framework natively predicts zero parity-violating birefringence in the infrared vacuum.*

7 Where Framework Signatures Live

Because the vacuum photon physics is topologically protected, observable framework-distinctive effects exist only in matter-coupled processes. We highlight the following domains:

- **Atomic parity violation:** Matter cells (octahedra) carry chiral fermions inheriting parity-violating structure from the substrate. This modifies the weak charge Q_W measured in atomic experiments (e.g., cesium, ytterbium [17,18]).
- **Anomalous magnetic moments:** Substrate-chirality contributions to $a_\mu = (g-2)/2$ offer a natural, geometric origin for the tensions observed at Fermilab [19].
- **Neutrino oscillations and CP violation:** The inherent chirality of the substrate will imprint distinct CP-violating amplitudes on MSW oscillations and heavy quark systems (LHCb, Belle II).
- **Optical activity in chiral metamaterials:** While vacuum entanglement is protected, entangled photons passed through engineered chiral topological media will exhibit specific phase modifications computable via the framework's local topological invariants.

8 Discussion and Conclusions

The neuro-symbolic framework's photon sector has reached maturity. Massless photons are not external *ad hoc* additions; they emerge spontaneously as fluctuations on the exact same entanglement bonds that drive the Q_3 matter crystallisation.

The chiral $\pi/4$ phases on the emergent line graph elegantly balance two distinct regimes: they provide **local topological photonics** (multi-band Chern number $C_{\mathcal{S}_7} = -1$, broken TR symmetry, linear band-touching nodes) at the microscopic scale, while exact geometric holonomies enforce **global IR protection**, shielding the continuum limit and unconditionally recovering the standard parity-conserving QED vacuum.

A tantalising direction for immediate future work lies in the derivation of the fine-structure constant. Structural analysis of the gauge-matter subspace indicates a phase space restricted to exactly 136 internal microstates plus 1 emission channel, strongly hinting at a purely combinatorial geometric derivation of $\alpha \approx 1/137$. Expanding the rotor-based photons in the annealing codebase to formalise this limit remains the primary focus of ongoing research.

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