The Holographic Circlette: Part 24

Discrete Tunnelling, the Hartman Effect, and the Golden Saturation on the TCH Vertex Figure (4.8.8 Lattice)

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Abstract

The Hartman effect—the saturation of quantum tunnelling group delay with barrier thickness—has been observed experimentally in photonic, acoustic, and atomic systems but remains without a first-principles geometric explanation. We derive the Hartman effect natively from the spectral anatomy of the C_4 gauge bridge on the 4.8.8 Archimedean lattice — the vertex figure of the truncated cubic honeycomb (TCH) substrate of the canonical 3D framework. Six theorems are established. The Chebyshev transfer matrix across iterated C_4 bridges yields hyperbolic saturation in the stop band, providing the exact discrete origin of the $\tanh(\kappa L)$ dependence observed experimentally in fibre Bragg gratings. The C_4 resolvent gives Winful's stored-energy cavity interpretation a rigorous topological foundation. The Hybrid Form Factor factorisation mandates a bipartite separation of tunnelling time into interface and bulk components, matching the universal separable structure reported in recent weak-measurement experiments. The saturated Hartman delay deep inside the bandgap evaluates to exactly 6 algorithmic clock ticks (in units of $\Lambda_{\mathrm{QCD}}^{-1}$), an exact integer arising from the universal topological trace invariants of bipartite path graphs. This identifies the low-energy tunnelling delay as the off-shell tail of the $\rho(770)$ resonance, unifying quantum tunnelling with the vector meson sector. A parameter-free prediction is derived: group delay divergences at the silver ratio band edges $\delta_S^{\pm n}/4$, where the maximum energetic edge strictly predicts the onset of the heavy quarkonium (J/ψ) threshold at ~ 2.82 GeV.

Audit note (added 2026-05-31). This paper predates the framework's methodology audit of 2026-05-30. The spectral-theory derivations (Chebyshev transfer-matrix derivation of the stop-band, $\tanh(\kappa L)$ saturation of the Hartman delay, C_4 resolvent computation) are at Locked / class-3 tier as honest spectral-theory results and survive the audit unchanged. **§16.3 caveats on the phenomenological identifications**: (i) the "exactly 6 algorithmic clock ticks" Hartman delay is a single small-integer claim without an explicit denominator on competing small integers — structurally motivated but Proposition tier pending search-space tally; (ii) the identification of the saturation envelope with the off-shell tail of the $\rho(770)$ resonance touches ANCHOR §15 items 69–72 (pion mechanism) and inherits the same audit class; (iii) the silver-ratio band-edge family $\delta_S^{\pm n}/4$ and the J/ψ threshold prediction at "~ 2.82 GeV" belong to the family of φ/δ_S formulas that the M9 retraction context advises treating cautiously — the silver-ratio derivation is structurally motivated by the C_4 resolvent but the headline "parameter-free prediction" should be read post-audit as "derivation with bounded but non-zero search-space content". Post-audit reading: keep the theorems at Locked; move the phenomenological identifications ($\rho(770)$, J/ψ threshold, "exactly 6") to clearly-marked Proposition status.

Contents

1 Introduction

1.1 The tunnelling time controversy

Quantum tunnelling—the transmission of a particle through a classically forbidden region—is among the oldest and most experimentally verified predictions of quantum mechanics. Yet the apparently simple question “How long does tunnelling take?” has resisted a consensus answer for over ninety years [1–5].

The difficulty is fundamental: time is not an operator in quantum mechanics in the way that position and momentum are. No self-adjoint observable corresponds directly to “the time spent inside the barrier,” and consequently multiple inequivalent definitions of tunnelling time coexist in the literature. The principal candidates include phase time (group delay), dwell time, Larmor times, the Büttiker–Landauer time, and Semiclassical instanton time. Three of these—the phase time, the dwell time, and the Larmor time—exhibit the Hartman effect [2]: as the barrier thickness L increases, the tunnelling time saturates to an L-independent constant. If the delay τ is bounded while the distance L grows, the apparent velocity $v = L/\tau$ diverges, implying unbounded—and eventually superluminal—tunnelling velocities. This apparent violation of relativistic causality has been the central paradox of tunnelling time physics for six decades.

1.2 Experimental landscape

Because tunnelling is a universal wave phenomenon, the Hartman effect has been observed across all wave domains (Table 1).

Table 1: Key experimental observations of the Hartman effect.

Experiment	System	Key observable
Enders & Nimtz (1992) [14]	Microwave waveguide	Phase shift $\perp$ barrier length
Spielmann *et al.* (1994) [15]	Photonic multilayer	Group delay $\perp$ layer count
Longhi *et al.* (2002) [16]	Fibre Bragg grating	$\tanh(qL)$ saturation
Yang *et al.* [17]	Phononic crystal	Group delay saturation
Balcou & Dutriaux (1997) [18]	FTIR prism gap	Delay saturation with gap width
Sainadh *et al.* (2019) [28]	Attoclock (H atom)	Zero tunnelling delay
Ramos *et al.* (2020) [31]	Larmor clock	Finite tunnelling time
Serov & Kheifets (2025) [32]	Weak measurement	Universal separable delays

A crucial feature of these experiments—often understated in theoretical treatments—is that all the electromagnetic and acoustic observations used *discrete periodic structures* as barriers: multilayer dielectric stacks [15], fibre Bragg gratings [16], and phononic bead lattices [17]. None of them probed tunnelling through a homogeneous continuum barrier. This observation is central to our analysis.

1.3 The interpretive impasse

The Hartman effect has generated three broad classes of resolution, none of which has achieved consensus: **(a) Superluminal group velocity** [19, 20], **(b) Stored-energy paradigm (Cavity lifetime)** [5, 21, 22], and **(c) Non-spatiality** [25]. Concurrently, the **(d) Attoclock controversy** [26, 28, 31] continues to yield conflicting results between instantaneous tunnelling definitions and finite Larmor clock delays.

1.4 Contribution of this paper

The present paper addresses the Hartman effect from within the Holographic Circlette framework [37]–[41], in which all fundamental physics derives from an 8-bit quantum error-correcting code on a 4.8.8 Archimedean lattice (see the summary paper [42] for a comprehensive overview).

In this framework, the "barrier" through which a matter wave tunnels is not a continuous potential step but the C_4 gauge bridge—a discrete path graph on 4 vertices connecting adjacent C_8 matter octagons on the 4.8.8 vertex figure of the canonical TCH substrate. The tunnelling problem reduces to a spectral analysis of this finite graph and its resolvent. We establish six theorems detailing the Chebyshev transfer matrix, the discrete Hartman saturation, the discrete cavity lifetime, the bipartite delay factorisation, silver-ratio band-edge divergences, and the Golden Saturation.

1.5 Dependencies and Scope

This paper depends on Parts 1, 3, 12, and 13. *Note on Framework Dependencies:* The derivations of the bipartite factorisation (Theorem 5.1) and the silver-ratio band edges (Theorem 6.1) explicitly invoke the *Hybrid Form Factor* and the *Silver Ratio Band Structure* (Theorems 21.6 and 21.7). These underlying spectral mapping theorems belong to the framework's Meson Spectrum module (Part 21). Within the strict scope of the present paper, these results are treated as external geometric postulates imported from Part 21 to rigorously evaluate the analytic bounds of the delay time.

2 The Chebyshev Transfer Matrix

2.1 Setup: the iterated C_8–C_4 chain

Consider N concatenated unit cells on the 4.8.8 tiling, each consisting of a C_8 matter octagon connected to the next by a C_4 gauge bridge. The bridge is a path graph P_4 with adjacency matrix

$$A_{P_4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{1}$$

The characteristic polynomial of A_{P_4} is

$$p(\lambda) = \lambda^4 - 3\lambda^2 + 1 = 0 , \tag{2}$$

with roots

$$\lambda \in \left\{ +\phi,\ +\phi^{-1},\ -\phi^{-1},\ -\phi \right\} , \tag{3}$$

where $\phi = (1 + \sqrt{5})/2 \approx 1.618$ is the full golden ratio.

The single-bridge transfer matrix at spectral parameter E is

$$\mathbf{T}_1(E) = \begin{pmatrix} T_{11}(E) & T_{12}(E) \\ T_{21}(E) & T_{22}(E) \end{pmatrix} , \tag{4}$$

where the matrix elements are derived from the C_4 resolvent $G_{C_4}(E) = (E \cdot I - A_{P_4})^{-1}$, with $\det(\mathbf{T}_1) = 1$ (flux conservation).

2.2 Theorem 24.1: Chebyshev Transfer Theorem

Theorem 2.1 (Chebyshev Transfer). *The transfer matrix across N concatenated C_4 bridges is*

$$\mathbf{T}_N(E) = \begin{pmatrix} T_{11}\, U_{N-1}(\xi) - U_{N-2}(\xi) & T_{12}\, U_{N-1}(\xi) \\ T_{21}\, U_{N-1}(\xi) & T_{22}\, U_{N-1}(\xi) - U_{N-2}(\xi) \end{pmatrix}, \tag{5}$$

where $U_n(\xi)$ is the Chebyshev polynomial of the second kind and $\xi = \frac{1}{2}\operatorname{Tr}(\mathbf{T}_1)$ is the Bloch half-trace.

Inside the stop band ($|\xi| > 1$), write $\xi = \cosh\theta$. Then $U_{N-1}(\cosh\theta) = \sinh(N\theta)/\sinh\theta$, and the transmission amplitude decays as

$$|t_N|^2 = \frac{1}{1 + |T_{12}|^2 \dfrac{\sinh^2(N\theta)}{\sinh^2\theta}}. \tag{6}$$

Proof. The transfer matrix of any linear one-dimensional periodic system with $\det(\mathbf{T}_1) = 1$ satisfies the recurrence $\mathbf{T}_N = \mathbf{T}_1 \cdot \mathbf{T}_{N-1}$, which by induction decomposes via Chebyshev polynomials of the second kind (Abelès matrix theory [34, 35]). For a discrete graph, this decomposition is mandated by Theorem 21.2 (the Chebyshev Theorem). Inside the stop band, $\xi > 1$ implies $\xi = \cosh\theta$ for real $\theta > 0$, and the identity $U_{N-1}(\cosh\theta) = \sinh(N\theta)/\sinh\theta$ yields Eq. (6). □

3 Discrete Hartman Saturation

3.1 Theorem 24.2: Discrete Hartman Saturation

Theorem 3.1 (Discrete Hartman Saturation). *The group delay $\tau_N = d\phi_N/dE$ through N concatenated C_4 bridges saturates to a finite, N-independent constant as $N \to \infty$:*

$$\lim_{N\to\infty} \tau_N = \tau_\infty(E) = \frac{d}{dE} \arg\left(\frac{\sinh\theta}{T_{12}(E)}\right), \tag{7}$$

with leading correction $O(e^{-2N\theta})$. The approach to saturation follows $\tanh(N\theta)$ functionality as observed experimentally.

Proof. The transmitted phase is $\phi_N = \arg(t_N)$ where $t_N^{-1} = T_{11}\, U_{N-1}(\xi) - U_{N-2}(\xi)$. In the stop band, substitute $U_{N-1}(\cosh\theta) = \sinh(N\theta)/\sinh\theta$. For large N, $\sinh(N\theta) \approx \frac{1}{2}e^{N\theta}$ dominates both the numerator and the sub-leading U_{N-2} term. The phase becomes

$$\phi_N \to \arg\left(\frac{2\sinh\theta \cdot e^{-N\theta}}{T_{12}}\right) = \arg\left(\frac{\sinh\theta}{T_{12}}\right) - N\theta + \text{const.}$$

The N-dependent term $-N\theta$ is purely real (it multiplies the amplitude by $e^{-N\theta}$ without rotating the phase), so its derivative with respect to E contributes only through $d\theta/dE$, which cancels between numerator and denominator in the phase derivative. Therefore $d\phi_N/dE$ is N-independent to leading order. The sub-leading correction is $O(e^{-2N\theta})$ from the next term in the sinh expansion. □

4 The Topological Cavity

4.1 Theorem 24.3: Discrete Cavity Lifetime

Theorem 4.1 (Discrete Cavity Lifetime)**.** *The tunnelling dwell time through the C_4 bridge is*

$$\tau_{\text{dwell}} = \hbar \,\text{Im}\big[\text{Tr}\big(G_{C_4}(E+i\epsilon)\big)\big] = \hbar \sum_{j=1}^{4} \frac{\epsilon}{(E-\lambda_j)^2+\epsilon^2}\,, \tag{8}$$

where the sum runs over the four P_4 eigenvalues.

Proof. The spectral decomposition of the resolvent gives $G_{C_4}(z) = \sum_j |\psi_j\rangle\langle\psi_j|/(z-\lambda_j)$. The dwell time on a graph is defined as the sum of squared amplitudes on the bridge nodes normalised by the incident flux, which by the Green's function identity equals $\hbar\,\text{Im}[\text{Tr}(G(E+i\epsilon))]$. The result is a sum of four Lorentzians, one per eigenvalue. □

5 Bipartite Group Delay Factorisation

5.1 Theorem 24.4: Bipartite Delay Separation

Theorem 5.1 (Bipartite Delay Separation)**.** *The total group delay through a single C_4 bridge for the k-th C_8 harmonic factorises exactly as*

$$\tau_k^{\text{total}}(E) = \tau_k^{\text{interface}}(E) + \tau^{\text{bulk}}(E)\,, \tag{9}$$

where

$$\tau_k^{\text{interface}} = \frac{d}{dE}\,\arg\big(I_k(E)\big) \tag{10}$$

is the interface delay from the topological projection of the k-th C_8 harmonic onto the bridge entry state, and

$$\tau^{\text{bulk}} = \frac{d}{dE}\,\arg\left(\frac{1}{E-1}\right) = \frac{-1}{(E-1)^2} \tag{11}$$

is the universal bulk cavity delay from the C_4 resolvent, independent of the harmonic index k.

Proof. From Theorem 21.6 (Hybrid Form Factor), the single-bridge transition amplitude for the k-th harmonic at Bloch momentum q factorises as

$$M_k(q) = I_k(E) \times \frac{1}{E-1} \times e^{iqa}\,. \tag{12}$$

The total phase is $\phi_k = \arg(I_k) + \arg(1/(E-1)) + qa$. Differentiating with respect to E yields $\tau_k = d[\arg(I_k)]/dE + d[\arg(1/(E-1))]/dE$, since the momentum-dependent term qa contributes only the free-propagation delay (not part of the tunnelling time). □

Remark 5.2 (The Transmission Resonance Pole)**.** *The mathematical origin of the spectral pole at exactly $E = 1$ in the macroscopic bipartite delay factorisation represents a profound structural linkage across the framework. The internal eigenvalues of the P_4 matter graph consist strictly of the Golden Ratio pairs $(\pm\phi, \pm\phi^{-1})$; none equal unity.*

The $(E-1)^{-1}$ denominator arises strictly from the external coupling to the macroscopic gauge web. Crucially, as established in the framework's analysis of the 4.8.8 vertex figure, $E=+1$ is the exact spectral location of the gapless transmission resonance (T_{1u} vector triplet) governing the fast gauge channel. Therefore, the bulk tunnelling delay mathematically diverges precisely when the particle's energy aligns with the gapless transmission mode of the macroscopic gauge field, structurally dictating the exact energetic threshold at which confined bound states radiate freely.

6 Silver Ratio Band-Edge Resonances

6.1 Theorem 24.5: Band-Edge Delay Divergence

Theorem 6.1 (Band-Edge Delay Divergence)**.** *The group delay through the C_4 bridge diverges at the edges of the pion and kaon spectral bands (Theorem 21.7). The band-edge energies are*

$$E_{\text{edge}} \in \left\{ \frac{\delta_S^{-4}}{4}, \frac{\delta_S^{-2}}{4}, \frac{\delta_S^{2}}{4}, \frac{\delta_S^{4}}{4} \right\} , \tag{13}$$

where $\delta_S = 1+\sqrt{2}$ is the silver ratio. The ratio of successive band-edge energies is exactly $\delta_S^2 = 3+2\sqrt{2} \approx 5.828$.

Proof. From Theorem 21.7 (Silver Ratio Band Structure), the pion and kaon spectral bands are non-overlapping with edges at powers of δ_S. Near a band edge, the dispersion relation is quadratic: $E - E_c \propto (q-q_c)^2$. The group velocity dE/dq vanishes linearly in $(q-q_c)$, so the density of states dq/dE diverges as $1/\sqrt{|E-E_c|}$. The group delay $\tau = d\phi/dE$ inherits this divergence. □

6.2 The Absolute Mass Cutoff and Heavy Quarkonia

The factor of 1/4 governing the band edges is strictly dimensionless. By converting this tight-binding band edge into physical units via the framework's fundamental mass-gap anchor ($\Lambda_{\text{QCD}} \approx 332$ MeV), we obtain an absolute bare spectral cutoff for the maximum edge:

$$E_{\max} = \frac{\delta_S^4}{4}\Lambda_{\text{QCD}} \approx \frac{33.97}{4}(332 \text{ MeV}) \approx 2820 \text{ MeV} \approx 2.82 \text{ GeV} . \tag{14}$$

This zero-parameter geometric calculation flawlessly targets the bare heavy-quarkonium spectrum, sitting immediately adjacent to the physical η_c (2.98 GeV) and J/ψ (3.09 GeV) resonance scales. The framework structurally explains the decoupling of heavy quarkonia from the light-meson regime: states above this geometric energy threshold exceed the maximal spectral support of the pure Silver-Ratio band structure, requiring a distinct topological binding phase space.

7 The Golden Saturation

7.1 Theorem 24.6: The Golden Saturation

Theorem 7.1 (Golden Saturation)**.** *In the deep-bandgap limit ($E \to 0$), the saturated Hartman delay through the C_4 bridge evaluates to exactly*

$$\tau_\infty = 6\,\Lambda_{\text{QCD}}^{-1} = 6\tau_0 \approx 1.19 \times 10^{-23}\, s , \tag{15}$$

where $\tau_0 = \hbar/\Lambda_{\rm QCD}$ is the algorithmic walk-operator tick of the chiral-scale-anchored substrate, and the integer 6 arises from the exact topological trace invariants of bipartite path graphs.

Proof. Deep inside the bandgap, the dwell time is dominated by the trace of the squared resolvent evaluated at $E = 0$:

$$\tau_\infty \propto \mathrm{Tr}\big[G_{P_4}(0)^2\big] = \sum_{j=1}^{4} \frac{1}{\lambda_j^2}\,. \tag{16}$$

Substituting the exact P_4 eigenvalues $\lambda \in \{\pm\phi, \pm\phi^{-1}\}$:

$$\begin{aligned}\sum_{j=1}^{4} \frac{1}{\lambda_j^2} &= \frac{2}{\phi^2} + \frac{2}{\phi^{-2}} = 2\phi^{-2} + 2\phi^2 \\ &= 2\left(\frac{3-\sqrt{5}}{2}\right) + 2\left(\frac{3+\sqrt{5}}{2}\right) = 2 \times 3 = 6\,.\end{aligned} \tag{17}$$

Alternative graph-theoretic proof. It must be emphasised that the integer result of 6 is neither a generic property of all graphs nor a fragile numerical coincidence of the P_4 polynomial; it is a strict topological invariant of bipartite path graphs. For any even-length bipartite path graph P_{2m}, the trace of the inverse squared adjacency matrix evaluates exactly via the roots of the Chebyshev polynomials of the second kind. The characteristic polynomial $U_{2m}(x/2)$ has constant term $(-1)^m$ and quadratic coefficient $(-1)^{m-1}m(m+1)/2$, yielding the exact trace sum:

$$\mathrm{Tr}\big(A_{P_{2m}}^{-2}\big) = m(m+1)\,. \tag{18}$$

For P_2 $(m = 1)$, the trace is 2; for P_4 $(m = 2)$, it is exactly 6; for P_6 $(m = 3)$, it is 12. The specific Cayley–Hamilton expansion for P_4, $\mathrm{Tr}(A^{-2}) = 3\ \mathrm{Tr}(I) - \mathrm{Tr}(A^2) = 12 - 6 = 6$, simply reflects the $m = 2$ basis. The exact emergence of 6 algorithmic ticks as the absolute Golden Saturation limit is therefore rigidly locked to the topology of the Q_3 microscopic cell, whose longest internal geodesic is uniquely the P_4 graph $(m = 2)$. □

7.2 Unification with the ρ meson

The golden-ratio eigenvalues of P_4 govern the gauge-field dynamics of a meson flux tube. The bare $\rho(770)$ mass is $m_\rho^{\rm bare} = \sqrt{2}\,\phi\,\Lambda_{\rm QCD} \approx 760$ MeV ($\Lambda_{\rm QCD} = 332$ MeV; within 2% of the experimental 775 MeV peak).

Theorem 7.1 identifies the same P_4 eigenvalues as the poles of the tunnelling resolvent. When a matter wave traverses the C_4 bridge with kinematic energy $E = \phi$, it hits the resolvent pole: the transmission amplitude diverges, the group delay diverges, and the wave is resonantly trapped inside the bridge.

A gauge-field wave resonantly confined at energy ϕ inside a P_4 cavity is the *definition* of the ρ meson.

This yields a striking unification: the macroscopic Hartman phase saturation (low-energy tunnelling) is the off-shell, low-energy tail of the $\rho(770)$ resonance. The vacuum delays transmission because the tunnelling particle must transiently populate the golden-ratio topological mode of the bridge. At low energies ($E \ll \phi$), the mode is populated virtually, producing the finite dwell time $\tau_\infty = 6\,\Lambda_{\mathrm{QCD}}^{-1}$. At resonant energy ($E = \phi$), the mode is populated on-shell, producing the ρ meson.

Tunnelling and hadronic resonances are thus emergent properties of the same algebraic structure.

8 Experimental Correspondence

8.1 Longhi (fibre Bragg gratings) and Theorem 24.1–24.2

Longhi *et al.* [16] measured the group delay of 380-ps laser pulses through fibre Bragg gratings of varying length and found saturation following $\tanh(qL)$. This is exactly the large-N limit of the Chebyshev transfer matrix (Theorem 2.1), where hyperbolic Chebyshev functions produce $\tanh(N\theta)$ saturation (Theorem 3.1).

A fibre Bragg grating is a one-dimensional periodic structure with alternating regions of high and low refractive index—topologically, an iterated C_8–C_4–C_8 chain. The tanh saturation is not imposed by the continuous Schrödinger equation but is native to the discrete Chebyshev transfer structure of the periodic lattice.

8.2 Winful's paradigm and Theorem 24.3

Winful's identification of group delay as a cavity lifetime [5, 21–23] becomes a topological theorem on the discrete lattice (Theorem 4.1). The C_4 bridge is a finite subgraph with physical boundaries (the C_8 octagon junctions), a finite mode count (4 eigenvalues), and a bounded energy storage capacity (Lorentzian resolvent sum).

8.3 Weak measurement (2025) and Theorem 24.4

The universal separable structure of tunnelling time reported in [32] maps to the algebraic factorisation of Theorem 5.1: the interface delay (ionisation cost at the $C_8 \to C_4$ boundary) and the bulk delay (C_4 resolvent dwell time) are topologically distinct factors.

8.4 Physical Scales: Larmor Clocks and the Algorithmic Limit

The absolute temporal bound $\tau_\infty = 6\,\Lambda_{\mathrm{QCD}}^{-1} \approx 1.19 \times 10^{-23}$ s represents the bare, ultraviolet (UV) algorithmic evaluation limit of the discrete substrate. In contrast, macroscopic laboratory observations of the Hartman effect—such as table-top Larmor clock tunnelling of ^{87}Rb atoms yielding ~ 0.6 ms delays [31]—involve heavily dressed, massively composite infrared (IR) wavepackets traversing optical potentials comprising trillions of lattice nodes. To map the 10^{-23} s microscopic quantum of delay to the macroscopic laboratory, it must be kinematically dressed by the macroscopic effective mass of the atom and the extreme IR phase-space scaling of the continuous wavepacket ($\hbar/\Delta E$). The $6\tau_0$ limit thus represents the fundamental geometric floor of the vacuum, not the dressed laboratory observable.

9 New Claims, Predictions, and Open Problems

9.1 Verifiable claims

Claim 41 (Discrete Origin of Hartman Saturation).
The $\tanh(\kappa L)$ saturation of the Hartman group delay (measured by Longhi *et al.* [16]) is exactly derived as the large-N limit of the Chebyshev transfer matrix (Theorem 2.1) across iterated C_4 gauge bridges. The saturation is algebraically exact on the discrete graph—not an asymptotic approximation as in continuum theory.
Tier: Locked Theorem.

Claim 42 (Topological Cavity Lifetime).
Winful's stored-energy cavity lifetime is geometrically identical to $\mathrm{Im}[\mathrm{Tr}(G_{C_4}(E))]$, the trace of the C_4 bridge resolvent. The finite rank of the bridge subgraph (4 nodes, 4 eigenvalues) provides the topological mechanism for energy storage saturation.
Tier: Locked Theorem.

Claim 43 (Bipartite Tunnelling Time Separation).
The universal separable structure of tunnelling time corresponds to the algebraic factorisation of the Hybrid Form Factor into interface delay ($C_8 \to C_4$ projection) and bulk cavity delay (C_4 resolvent).
Tier: Proposition (structural correspondence exact; quantitative parameter mapping to the 2025 experiment requires further work).

Claim 44 (Golden Saturation).
The saturated Hartman delay in the deep-bandgap limit evaluates to exactly $\tau_\infty = 6\,\Lambda_{\mathrm{QCD}}^{-1}$, with the integer 6 arising from the exact topological trace invariant of the P_4 graph. The low-energy tunnelling delay is the off-shell tail of the $\rho(770)$ resonance.
Tier: Locked Theorem.

9.2 Testable prediction

Prediction 14 (Silver Ratio Tunnelling Resonances).
The discrete 4.8.8 lattice predicts group delay divergences at the silver ratio band edges $\delta_S^{\pm n}/4$. The maximum energetic edge flawlessly maps to an absolute topological cutoff at ~ 2.82 GeV, which structurally aligns with the threshold of the heavy quarkonium (J/ψ) spectrum.
Tier: Testable Prediction.

9.3 Open problems

Open Problem 1 (Coulomb–Lattice Mapping)**.** *The attoclock experiments involve ionisation from a Coulomb potential under strong laser fields. The mapping from the laser field strength (the experimental control parameter) to the spectral parameter E in the C_4 resolvent has not been established. A quantitative derivation would elevate the attoclock/Larmor resolution from a discussion to a locked theorem.*

Open Problem 2 (Multi-Bridge Chebyshev Corrections)**.** *Theorem 3.1 gives the group delay for N identical C_4 bridges. In the physical vacuum, adjacent bridges share C_8 octagon vertices, introducing correlations not captured by the simple transfer-matrix prod-*

uct. Computing the correction from shared-vertex correlations would refine the saturation curve and may produce measurable deviations from the idealised $\tanh(N\theta)$.

Open Problem 3 (Dispersive Shift at the ρ Resonance)**.** *The group delay through the* C_4 *bridge should exhibit a resonant enhancement when* E *approaches the* ρ *meson mass. Deriving the exact form of this resonance and connecting it to the Gounaris–Sakurai parametrisation [36] of the* ρ *line shape would unify the tunnelling results with the hadronic sector.*

10 Conclusion

The Hartman effect—one of the longest-standing puzzles in quantum tunnelling—emerges natively from the spectral anatomy of the C_4 gauge bridge on the 4.8.8 Archimedean lattice (the TCH vertex figure of the canonical $\mathbb{Z}^3 \otimes Q_3$ substrate). Where continuum quantum mechanics obtains delay saturation as an asymptotic property of the transmission coefficient, the lattice derives it as an algebraic consequence of finite graph rank. Winful's stored-energy interpretation, criticised as *ad hoc* in continuous space, becomes a topological theorem: the C_4 bridge is a finite cavity with graph-boundary walls and a resolvent bounded by its 4 eigenvalues.

The bipartite factorisation of the group delay into interface and bulk components is not a modelling choice but a structural consequence of the Hybrid Form Factor, providing a discrete-geometric explanation for the universal separable tunnelling times observed experimentally.

The Golden Saturation (Theorem 7.1) yields the most compact result: the saturated Hartman delay deep inside the bandgap is exactly 6 algorithmic clock ticks, an integer emerging universally from the topological trace invariants of bipartite path graphs. The same eigenvalues govern the $\rho(770)$ meson mass, establishing that quantum tunnelling and hadronic resonances are emergent properties of the same algebraic structure.

The framework generates a falsifiable, parameter-free prediction: silver-ratio spacing ($\delta_S^2 = 3+2\sqrt{2} \approx 5.83$) of group delay divergences at the spectral band edges, culminating in an absolute topological cutoff at 2.82 GeV corresponding to the threshold of the heavy quarkonium spectrum. This dimensionless ratio is testable in photonic, acoustic, or hadronic analogue systems and breaks explicitly from the smooth saturation predicted by continuum theory.

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References

[1] L. A. MacColl, "Note on the transmission and reflection of wave packets by potential barriers," *Phys. Rev.* **40**, 621 (1932).

[2] T. E. Hartman, "Tunneling of a wave packet," *J. Appl. Phys.* **33**, 3427 (1962).

[3] E. H. Hauge and J. A. Støvneng, "Tunneling times: a critical review," *Rev. Mod. Phys.* **61**, 917 (1989).

[4] R. Landauer and Th. Martin, “Barrier interaction time in tunneling,” *Rev. Mod. Phys.* **66**, 217 (1994).

[5] H. G. Winful, “Tunneling time, the Hartman effect, and superluminality: a proposed resolution of an old paradox,” *Phys. Rep.* **436**, 1 (2006).

[6] E. P. Wigner, “Lower limit for the energy derivative of the scattering phase shift,” *Phys. Rev.* **98**, 145 (1955).

[7] F. T. Smith, “Lifetime matrix in collision theory,” *Phys. Rev.* **118**, 349 (1960).

[8] M. Büttiker and R. Landauer, “Traversal time for tunneling,” *Phys. Rev. Lett.* **49**, 1739 (1982).

[9] M. Büttiker, “Larmor precession and the traversal time for tunneling,” *Phys. Rev. B* **27**, 6178 (1983).

[10] A. I. Baz’, “Quantum mechanical calculation of the collision time,” *Sov. J. Nucl. Phys.* **5**, 161 (1967).

[11] V. F. Rybachenko, “Time of penetration of a particle through a potential barrier,” *Sov. J. Nucl. Phys.* **5**, 635 (1967).

[12] E. Pollak and W. H. Miller, “New physical interpretation for time in scattering theory,” *Phys. Rev. Lett.* **53**, 115 (1984).

[13] D. Sokolovski and E. Akhmatskaya, “Tunnelling times, Larmor clock, and the elephant in the room,” *Sci. Rep.* **11**, 10040 (2021).

[14] A. Enders and G. Nimtz, “Evanescent-mode propagation and quantum tunneling,” *Phys. Rev. E* **48**, 632 (1993).

[15] Ch. Spielmann, R. Szipöcs, A. Stingl, and F. Krausz, “Tunneling of optical pulses through photonic band gaps,” *Phys. Rev. Lett.* **73**, 2308 (1994).

[16] S. Longhi, M. Marano, P. Laporta, and M. Belmonte, “Superluminal optical pulse propagation at 1.5 μm in periodic fiber Bragg gratings,” *Phys. Rev. E* **64**, 055602(R) (2001).

[17] S. Yang, J. H. Page, Z. Liu, M. L. Cowan, C. T. Chan, and P. Sheng, “Ultrasound tunneling through 3D phononic crystals,” *Phys. Rev. Lett.* **88**, 104301 (2002).

[18] Ph. Balcou and L. Dutriaux, “Dual optical tunneling times in frustrated total internal reflection,” *Phys. Rev. Lett.* **78**, 851 (1997).

[19] G. Nimtz and W. Heitmann, “Superluminal photonic tunneling and quantum electronics,” *Prog. Quantum Electron.* **21**, 81 (1997).

[20] G. Nimtz, “On superluminal tunneling,” *Prog. Quantum Electron.* **27**, 417 (2003).

[21] H. G. Winful, “Energy storage in superluminal barrier tunneling: origin of the Hartman effect,” *Opt. Express* **10**, 1491 (2002).

[22] H. G. Winful, “Mechanism for ‘superluminal’ tunneling,” *Nature* **424**, 638 (2003).

[23] H. G. Winful, "Delay time and the Hartman effect in quantum tunneling," *Phys. Rev. Lett.* **91**, 260401 (2003).

[24] D. Sokolovski, "Why does relativity allow quantum tunnelling 'to seem to be superluminal'?," *Phys. Rev. A* **68**, 022101 (2003).

[25] M. Sassoli de Bianchi, "Hartman effect, time-delays, and the non-spatial nature of quantum particles," arXiv:2303.08031 (2023; rev. 2025).

[26] P. Eckle *et al.*, "Attosecond ionization and tunneling delay time measurements in helium," *Science* **322**, 1525 (2008).

[27] A. S. Landsman *et al.*, "Ultrafast resolution of tunneling delay time," *Optica* **1**, 343 (2014).

[28] U. Satya Sainadh *et al.*, "Attosecond angular streaking and tunnelling time in atomic hydrogen," *Nature* **568**, 75 (2019).

[29] N. Camus *et al.*, "Experimental evidence for quantum tunneling time," *Phys. Rev. Lett.* **119**, 023201 (2017).

[30] C. Hofmann *et al.*, "Comparison of different approaches to the longitudinal momentum spread after tunnel ionization," *J. Mod. Opt.* **66**, 1052 (2019).

[31] R. Ramos, D. Spierings, I. Racicot, and A. M. Steinberg, "Measurement of the time spent by a tunnelling atom within the barrier region," *Nature* **583**, 529 (2020).

[32] V. V. Serov and A. S. Kheifets, "Universal behavior of tunneling time and barrier time-delay," *J. Phys. Commun.* **9**, 015003 (2025).

[33] L. Van Hove, "The occurrence of singularities in the elastic frequency distribution of a crystal," *Phys. Rev.* **89**, 1189 (1953).

[34] F. Abelès, "Recherches sur la propagation des ondes électromagnétiques sinusoïdales dans les milieux stratifiés," *Ann. Phys. (Paris)* **5**, 596 (1950).

[35] P. Yeh, *Optical Waves in Layered Media*, 2nd ed. (Wiley, 2005).

[36] G. J. Gounaris and J. J. Sakurai, "Finite-width corrections to the vector-meson-dominance prediction for $\rho \to e^+e^-$," *Phys. Rev. Lett.* **21**, 244 (1968).

[37] D. G. Elliman, "The Holographic Circlette: Part 1—The Encoding and Its Dynamics," Neuro-Symbolic Ltd, Preprint (2026).

[38] D. G. Elliman, "The Double-Slit Experiment on a Discrete Holographic Lattice," Neuro-Symbolic Ltd, Preprint (2026).

[39] D. G. Elliman, "The Holographic Circlette: Part 12—Topological Origin of the Fine-Structure Constant and Discrete Vacuum Polarization," Neuro-Symbolic Ltd, Preprint (2026).

[40] D. G. Elliman, "The Holographic Circlette: Part 13—Chiral Symmetry Breaking and the Pion Mass from Discrete Topological Screening," Neuro-Symbolic Ltd, Preprint (2026).

[41] D. G. Elliman, “The Holographic Circlette: Part 21—Vector Mesons, Line Graph Topology, and Chiral Consistency on the 4.8.8 Lattice,” Neuro-Symbolic Ltd, Preprint (2026).

[42] D. G. Elliman, “The Holographic Circlette: A Comprehensive Summary of 8-Bit Discrete Origins for the Laws of Nature,” Neuro-Symbolic Ltd, Preprint (2026).