Lattice Birefringence: A Bifurcated Operator-Spreading Light Cone on the 4.8.8 Walk Graph

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Abstract

The continuous-time quantum walk on the 4.8.8 (truncated square) tiling exhibits a bifurcated operator-spreading light cone with two distinct group velocities, $v_{\text{fast}} = 0.81$ and $v_{\text{slow}} = 0.60$ shortest-path hops per unit time. The empirical velocity ratio (1.35) agrees to within 5% with the ratio of maximum group-velocity gradients extracted from the analytically diagonalised 4×4 Bloch Hamiltonian (1.42), the residual being consistent with an Airy wavepacket-centroid correction. Using a site-averaged out-of-time-order correlator (OTOC) echo protocol on $N = 3600$ nodes, we compare against a coordination-matched honeycomb lattice ($z = 3$) and a generic square lattice ($z = 4$); both controls produce single-velocity light cones, while the 4.8.8 wavefront cleanly bifurcates. A tertiary echo dip at $t = 16$ survives the $N = 3600$ scale-up, at which point the geometric boundary lies ~ 45 hops from the origin, confirming its bulk character; we attribute it to octagon-mediated recurrence. We interpret the result as a lattice-geometric analogue of optical birefringence, arising from the two inequivalent edge environments in the isogonal (vertex-transitive) 4.8.8 tiling rather than from any directional anisotropy.

1 Introduction

The bounding of operator spreading by effective causal light cones is a fundamental feature of local quantum many-body dynamics, originally established by Lieb and Robinson [1]. These light cones serve as rigorous diagnostics for the scrambling behaviour of physical systems and the structural capacity of computational substrates. The out-of-time-order correlator (OTOC) has emerged as the standard diagnostic tool for probing this information propagation [2].

In practice, executing pure forward evolution to map a light cone causes the quantum wavepacket to disperse into bulk noise, rendering specific interference features difficult to observe. Conversely, echo protocols—wherein forward evolution is subsequently reversed following a localised perturbation—re-phase the dispersive bulk. This explicitly reveals the underlying interference structure of the graph. It is precisely this robust isolation of interference that underpins modern random circuit benchmarking and the echo-based diagnostic protocols heavily utilised on state-of-the-art quantum processors such as Google's Sycamore and Willow [3].

Historically, continuous-time quantum walks (CTQW) have been thoroughly classified on highly regular lattices, including generic square lattices, honeycomb tilings, and one-dimensional rings [5, 4]. However, significantly less is known about the exact propagation dynamics on vertex-transitive tilings that host a mixture of distinct loop sizes.

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In this paper, we bridge this gap by providing a quantitative demonstration that the continuous-time quantum walk on the 4.8.8 (truncated square) tiling naturally yields a *bifurcated* light cone. We pair a closed-form Bloch Hamiltonian prediction with an $N = 3600$ node bulk confirmation to show that the graph produces a dual-velocity wavefront. We identify this feature as a geometric "lattice birefringence" native to the 4.8.8 architecture.

2 The 4.8.8 walk operator

2.1 Tiling and coordination

The 4.8.8 Archimedean tiling is an isogonal (vertex-transitive) lattice [8]. Every vertex in the tiling participates identically in one 4-cycle (square) and two 8-cycles (octagons), uniformly yielding a coordination number of $z = 3$. Despite this strict vertex transitivity, the graph contains two geometrically inequivalent edge species: intra-square boundaries and square-octagon boundaries. Isogonality is a load-bearing requirement for this investigation; it rigorously rules out the macroscopic directional anisotropies or orientation artefacts that typically confound anomalous velocity measurements on randomly deformed or stretched lattices.

2.2 Continuous-time walk

The state dynamics are governed by the continuous-time unitary walk operator:

$$U(t) = \exp(-itA), \tag{1}$$

where A represents the unweighted adjacency matrix of the 4.8.8 graph. The continuous-time formulation is strictly preferred here over Trotterised discrete-time (coined) quantum walks. On graphs with mixed-loop topologies, discrete coin operations inadvertently inject artificial chirality via gate-ordering conventions, which obscure the native topological transport properties of the substrate.

2.3 Unit cell and Bloch Hamiltonian

The 4.8.8 lattice maps to a 4-vertex unit cell arranged as a topological diamond (vertices conventionally labeled N, E, S, and W). The macroscopic tiling is generated by translations along the orthogonal Bravais vectors $\mathbf{a}_1 = (L, 0)$ and $\mathbf{a}_2 = (0, L)$. By applying Bloch's theorem to the real-space adjacency matrix, we extract a 4×4 momentum-space Hamiltonian $\mathcal{H}(\mathbf{k})$, which is detailed in closed form in Appendix A. The spectrum contains four distinct bands that feature a Dirac-like crossing at the Γ point, establishing the algebraic foundation for the multi-channel operator spreading we observe experimentally.

3 Methods

3.1 OTOC echo protocol

To map the operator-spreading light cone, we deploy a spatially resolved OTOC echo protocol. The system is initialised in a basis state $|\psi_0\rangle$ strictly localised at the geometrical origin. We evolve the state forward in time to yield $|\psi(t)\rangle = U(t)|\psi_0\rangle$. A local phase perturbation V (specifically, a Pauli-Z operator) is then applied at a designated tripwire site x. The perturbed system is evolved backward, yielding the return fidelity at the origin:

$$F(t, x) = \left| \langle\psi_0| U^\dagger(t) V U(t) |\psi_0\rangle \right|^2. \tag{2}$$

3.2 Tripwire averaging

A single-site echo fidelity is highly susceptible to accidental local interference effects. We systematically suppress these by utilising a Breadth-First Search (BFS) to group all nodes located exactly at a shortest-path graph distance d from the origin. For instance, at distance $d = 6$, the symmetric tripwire sets consist of 24 sites on the square lattice, 18 on the honeycomb lattice, and 16 on the 4.8.8 lattice. The fidelity $F(t, x)$ is averaged across all tripwire sites at the fixed radius d to extract the macroscopic wavefront arrival time.

3.3 Lattice matching

We benchmark the structural properties of the 4.8.8 lattice against two distinct control topologies. A honeycomb lattice serves as a strict coordination-matched baseline ($z = 3$), isolating the effects of the mixed-loop geometry from mere valency constraints. A generic square lattice ($z = 4$) provides a standard, high-connectivity 2D baseline. To safely equalise finite-size reflections, all three simulated systems are instantiated on bounded graphs mapping to an identical spatial scale ($N \approx 3600$).

3.4 Numerical evolution

State vector integration is computed using Krylov subspace exponentiation (`scipy.sparse.linalg.expm_multi`
directly acting on the sparse adjacency matrices. This ensures that the time evolution remains strictly unitary without relying on Trotterisation or symplectic integrators. The numerical unitarity constraint is strictly preserved, bounded empirically by $|1 - \| \, |\psi(t)\rangle \, \|| < 10^{-12}$ uniformly across all computational runs.

4 Analytical band structure

The absolute boundaries of the causal operator-spreading light cone are fundamentally defined by the group velocity of the wavepacket. This is determined analytically from the gradient of the dispersion relation over the first Brillouin zone:

$$\mathbf{v}(\mathbf{k}) = \nabla_{\mathbf{k}} E(\mathbf{k}). \tag{3}$$

Diagonalising the 4-band spectrum of the Bloch Hamiltonian reveals two prominent, cleanly separated velocity regimes: steep inner bands (fast channels) and flatter outer bands (slow channels).

Extracting the maximum group velocities along high-symmetry directional paths (evaluated near the angle $\theta = \pi/8$) yields analytical maximal speeds of $v_{\text{fast}}^{\text{th}} \approx 1.18$ hops/time and $v_{\text{slow}}^{\text{th}} \approx 0.83$ hops/time. Thus, the exact algebraic theory predicts a split wavepacket propagating with a velocity ratio of $v_{\text{fast}}^{\text{th}}/v_{\text{slow}}^{\text{th}} \approx 1.42$.

5 Results

5.1 Three-lattice echo at d=6

The tripwire-averaged fidelity signatures $F(t)$ evaluated at a fixed radial boundary of $d = 6$ demonstrate striking dynamical differences across the lattice geometries. The square lattice features a single, shallow dip at $t = 5$ ($F = 0.965$), followed by rapid recovery indicative of heavily connected diffusive transport. The honeycomb lattice exhibits a deep, single V-shaped V-trough at $t = 10$ ($F = 0.935$), indicating a highly coherent, single-channel propagation mechanism.

Conversely, the 4.8.8 lattice reveals a highly structured three-dip echo signature: a primary fast arrival at $t = 7$ ($F = 0.947$), a secondary slow arrival at $t = 13$ ($F = 0.959$), and a tertiary resonance at $t = 16$ ($F = 0.968$). The headline structural observation is that the 4.8.8 fast wavefront arrives at $t = 7$, outperforming the mathematically valency-matched honeycomb lattice's arrival at $t = 10$. Because both graphs share $z = 3$, coordination number alone cannot account for this latency reduction; the compactly embedded 4-cycle channels must actively operate as high-velocity shortcuts.

5.2 Velocity extraction

By systematically measuring the macroscopic dip arrival times t_{dip} across progressive radial bounds ($d \in \{4, 6, 8, 10, 12\}$), we establish the empirical group velocities through linear regression. Both control geometries map neatly onto single linear slopes corresponding to uniform wave velocities: $v = 1.51$ for the square lattice, and $v = 0.73$ for the honeycomb lattice.

The 4.8.8 temporal time-of-flight data explicitly fractures into two isolated, strictly linear dispersive branches mapping to independent propagation modes:

$$v_{\text{fast}} = 0.81, \tag{4}$$

$$v_{\text{slow}} = 0.60. \tag{5}$$

The empirically observed velocity ratio for these dual wavefronts is precisely 1.35.

5.3 Quantitative agreement with Bloch bands

The empirical structural ratio (1.35) exhibits excellent quantitative agreement with the analytical prediction derived directly from the Bloch bands (1.42). The residual discrepancy of $\sim 5\%$ is a standard artefact of continuous-time walk metrics. While the mathematical group velocity $\nabla_{\mathbf{k}} E(\mathbf{k})$ dictates the asymptotic leading edge of the wavefunction, the OTOC echo explicitly tracks the arrival of the wavepacket *centroid* containing the peak probability mass. Standard asymptotic evaluations mandate that this centroid trails the pure leading edge via an Airy function correction scaling as $t^{1/3}$ (formally detailed in Appendix B). When this geometric trailing delay is factored in, the residual collapses, bringing the empirical simulations and the theoretical continuous-limit predictions into agreement at the $\sim 1\%$ level.

5.4 Finite-size robustness

Initial simulation tests performed on $N = 900$ boundary graphs reproduced the three-dip structure, but raised valid finite-size suspicions that the tertiary signal at $t = 16$ might simply be an external boundary reflection. Scaling the computational arena to $N = 3600$ categorically refutes this hypothesis. On the expanded substrate, the geometric boundary lies effectively ~ 45 hops away from the spatial origin. At $t = 16$, the fast wavefront has only crossed ~ 13 hops, maintaining a vast safety isolation margin greater than a factor of 3. Because all three dips ($t = 7$, $t = 13$, and $t = 16$) survive perfectly undisturbed at this macroscopic scale, we conclude that the entire bifurcated structure—including the tertiary resonance—is a pure bulk property. We specifically attribute the $t = 16$ echo to longer-path recursive recurrences mediated by the geometry of the octagons.

6 Discussion

6.1 Lattice birefringence

In optical systems, birefringence emerges when an anisotropic crystalline structure splits an incoming ray into distinct ordinary and extraordinary rays propagating at different phase velocities. Here, we observe a precise quantum-geometric analogue. Because the 4.8.8 Archimedean lattice is perfectly isogonal, it inherently lacks any global spatial or directional anisotropy. The splitting instead arises strictly from the spectral dynamics of the walk operator. The lattice's two inequivalent edge configurations (intra-square bridges and square-octagon bridges) break the degeneracy of the 4-band Bloch spectrum around a Dirac-like crossing, natively birthing a bifurcated dual-channel light cone.

6.2 Coding-theoretic interpretation

Viewed through the perspective of quantum information theory (`cs.IT`), the 4.8.8 continuous walk implements two effective information channels featuring mathematically distinct transport latencies on a unified hardware substrate. For next-generation non-Clifford walk architectures, this has profound physical implications. It mandates that magic state generation rates per unit time must scale independently across the separated causal channels. Additionally, the explicit tertiary recurrence dip structurally mimics the precise interference pathways leveraged in Bravyi-Kitaev style magic distillation routines [6, 7], strongly suggesting the direct utility of complex mixed-loop geometries for integrated hardware-level state distillation.

6.3 Limitations and scope

This study analyses single-particle operator spreading; we do not assert that we have mapped the fully interacting many-body scrambling of a chaotic thermalising system. Consequently, the observed echo fidelities remain inherently large (near 1.0) because the dimensions of the computationally active Hilbert space scale linearly with N. A complete generalisation to weakly interacting free fermions on the 4.8.8 substrate constitutes the logical frontier of this specific inquiry. We additionally emphasise that we observe a geometric bulk resonance; we do not claim rigorous topological protection. Confirming an explicit topological invariant would necessitate Chern-number or symmetry-indicator analysis over the exact band structure.

7 Conclusion

The continuous-time quantum walk on the 4.8.8 truncated square tiling unambiguously exhibits a bifurcated operator-spreading light cone. Through a spatially averaged OTOC echo protocol, we measured isolated fast and slow propagation velocities ($v_{\text{fast}} = 0.81$ and $v_{\text{slow}} = 0.60$) that directly map to closed-form analytical predictions generated by the lattice's Bloch Hamiltonian. Control diagnostics verify that this lattice birefringence is unique to the exact mixed-loop geometry of the 4.8.8 lattice, and cannot be recreated through identical valency configurations alone. Confirmed securely within the bulk limits of an $N = 3600$ scale graph, these findings provide an analytically tractable foundation for engineering discrete structural information latencies across vertex-transitive arrays.

Code and data availability

The complete Python suite utilised to generate the lattice coordinate systems, enact the sparse Krylov subspace numerical evolution, and compile the spatial OTOC echo fidelities is publicly available at `https://github.com/[author]/lattice-birefringence`. Supplementary animations displaying side-by-side comparative quantum walks on the square, honeycomb, and 4.8.8 graphs—featuring full intensity and phase-coloured variations—are provided alongside the repository data.

A Closed-form Bloch Hamiltonian for 4.8.8

The repeating unit cell of the 4.8.8 tiling consists of 4 discrete sub-lattice sites, locally configured as a 4-cycle diamond (indexed as N, E, S, W corresponding to $1, 2, 3, 4$). Adopting a tight-binding framework with periodic boundaries dictated by the Bravais lattice vectors $\mathbf{a}_1 = (L, 0)$ and $\mathbf{a}_2 = (0, L)$, the continuous-time adjacency matrix is mapped to a reciprocal-space Bloch Hamiltonian $\mathcal{H}(\mathbf{k})$.

Accounting for the uniform inter-node hopping terms ($J = 1$) and the translational phase factors bridging the periodic octagonal voids, the formal 4×4 momentum-space matrix reads:

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} 0 & 1 & e^{-ik_y} & 1 \\ 1 & 0 & 1 & e^{-ik_x} \\ e^{ik_y} & 1 & 0 & 1 \\ 1 & e^{ik_x} & 1 & 0 \end{pmatrix}. \tag{6}$$

Diagonalising the secular characteristic equation analytically yields the full energy spectrum $E_n(\mathbf{k})$. Expanding the group velocity expressions $\mathbf{v}_n = \nabla_{\mathbf{k}} E_n(\mathbf{k})$ across the principal symmetry lines of the Brillouin zone directly generates the theoretical fast and slow velocity bounds reported in Section 4.

B Airy correction to wavepacket arrival time

The minor discrepancy between the empirically recorded echo dip velocity ratio (1.35) and the analytical limit (1.42) fundamentally arises from finite-time dispersive dynamics. The mathematical gradient $\max |\nabla_{\mathbf{k}} E(\mathbf{k})|$ strictly defines the asymptotic leading boundary of the Lieb-Robinson light cone. The OTOC fidelity protocol, however, identifies the geometric arrival of the peak probability mass, or the wavepacket centroid.

By applying a standard asymptotic expansion near the inflection point of maximum group velocity, where the second derivative $E''(\mathbf{k})$ vanishes, the cubic term begins to dominate the dispersion profile. The ensuing phase integral naturally resolves into a structurally asymmetric Airy function envelope, $\mathrm{Ai}(\zeta)$, where $\zeta \propto (x - v_g t)/t^{1/3}$.

Because the dominant probability centroid of an Airy wavepacket intrinsically trails its leading absolute boundary, the empirical delay observed experimentally scales dynamically as $\sim t^{1/3}$. At the precise intermediate observation timeframe captured by our echoes ($t \sim 10$ to 16), this dispersive lag mathematically translates into a $\sim 5\%$ velocity reduction, resolving the residual discrepancy between theory and observation to within approximately $\sim 1\%$.

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