

# Gravity and fermions on a bond-centred lattice

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## Abstract

Can a lattice support the weak gravitational field and relativistic matter without introducing unstable or unwanted light modes? We examine two constructions on a lattice of cells centred on the bonds of a cubic grid. The first is a linearised gravitational action with continuous time. Its curvature and kinetic terms are taken from Regge calculus and the canonical formulation of general relativity. A condition on the lapse constraints removes an instability found in a predecessor action. For the resulting law, PL-1N, an exact finite certificate establishes leading long-wavelength constraint closure in every direction, conditional on the stated exact coefficients. Spectral tests find two massless transverse-traceless modes, one mode at the lattice scale, and no physical instability in the sampled region. These tests do not prove stability throughout the Brillouin zone. The Newtonian and stationary gravitomagnetic responses check imposed normalisations; the time-dependent source sector still differs from general relativity at finite spacing.

The second construction is a Wilson-type fermion action, PP+, on a separately assumed four-dimensional extension. Products of spinor projectors give one free light Dirac field and time-cell reflection-positivity evidence, with a conditional extension to U(1) and SU(3) gauge links. Hypercubic symmetry protects a common leading kinetic cone in the symmetric continuum description, while allowing species-dependent higher-order dispersion. The fermion's leading anisotropy coefficient is exactly  $+1/24$ . We illustrate how existing high-energy constraints would restrict a physical spacing if the free PP+ branch and its dispersion coefficient describe electrons. The two constructions establish properties of specified actions. Their combination into a consistent theory, and their derivation from quantum information, remain open.

## 1 Introduction

A discrete description of spacetime must account for more than the shape of its cells. It must explain how disturbances propagate, which disturbances represent particles, and which merely change the coordinates used to describe the same state. Gravity makes this distinction especially demanding. In general relativity (GR), the lapse and shift express freedom to choose time slices and spatial coordinates. Their equations constrain the physical initial data. If discretisation breaks these relations, a coordinate degree of freedom can acquire unwanted dynamics, including negative kinetic energy.

Regge calculus represents curvature by the failure of flat building blocks to fit around a common edge without an angular deficit [1]. It can reproduce the continuum weak-field theory [8], but this does not settle its behaviour at finite lattice spacing. Bahr and Dittrich showed how broken coordinate symmetries in curved Regge configurations become pseudo-constraints in the canonical description [5]. The constraint language used below is Dirac's [4].

Here the cells are oblate square bipyramids, each formed by joining two square pyramids at their bases. They sit on the bonds joining neighbouring cube centres. This geometry arose in the  $\mathbb{Z}^3 \otimes Q_3$  quantum-information framework, where  $Q_3$  names the cell. The notation does not assert a tensor factorisation of physical spacetime. The present paper asks a narrower question: what follows when explicit gravitational and fermionic actions are placed on this geometry? Quantum registers, their observable algebras and their records motivate the programme, but do not yet select either action.

For gravity we use a spatial lattice and continuous time, with a linearised Regge–ADM action for edge lengths [2, 3]. An earlier choice, PL-1, has unstable modes close to the face-diagonal directions of the cubic grid. We impose a condition on its lapse constraints, then choose the remaining normalisation to match the static GR response. The resulting law is PL-1N. We prove a leading-order constraint identity and examine its spectrum and source response. The finite sampling is extensive near the known difficult directions, but is not a proof of stability everywhere.

The fermion construction uses a different time prescription: a four-dimensional Euclidean lattice with equal steps in all directions. Its purpose is to test whether this geometry can carry a useful relativistic matter action. It is not obtained by quantising PL-1N, and its positivity result does not establish positivity of a combined matter–gravity theory.

**Results and their standing.** Each result is labelled with its standing; the labels are defined in §2.4.

1. A lemma: one lapse combination closes exactly at  $\mathbf{k}$  if and only if a scalar coupling  $c(\mathbf{k})$  vanishes. This is possible only where the momentum-constraint metric  $G^\dagger \mathcal{M} G$  is singular, which on this lattice occurs, in the sampled zone, only on the  $\langle 110 \rangle$  lines. The  $c = 0$  condition has a linear solution component  $L_1$  in the lapse weights (§3). [PROVED; LOCUS SAMPLED]
2. A theorem: at the closed-form lapse  $R^*$ , a nonzero lapse combination closes to  $O(k^2)$  along every direction  $\mathbf{u} \neq 0$  (Theorem 4). [PROVED, CONDITIONAL ON THE EXACT-COEFFICIENT IDENTIFICATION]
3. The spectrum: two massless transverse-traceless (TT) modes with  $c^2 = 1$ , one massive mode near  $\omega = 4$  in lattice units, and  $O(k^4)$  dispersion coefficients that are non-positive on every sampled ray (§4). [SAMPLED]
4. Stability: no growing or negative-energy mode at any sampled point of the Brillouin zone, including the  $\langle 110 \rangle$  neighbourhoods where PL-1 fails (§5). [SAMPLED, NOT PROVED]
5. Source couplings: under declared couplings the static response has an attractive isotropic  $1/k^2$  pole with no gravitational slip, and the stationary transverse response matches the GR gravitomagnetic kernel (§6). [CONSISTENCY CHECKS; FIXED BY THE NORMALISATION]
6. The longitudinal sector: the lattice breaks time reparametrisation explicitly at  $O(k^4)$ . At the tested points there is no additional light mode or negative residue. A breathing, mass-conserving source has a squared coupling to the graviton modes suppressed as  $(ka)^4$  per Fourier mode (§6.1). [EXPLORATORY]
7. The fermion sector: on the four-dimensional extension of the tiling, the finite parameter screen within the  $B_4$ -covariant nearest-neighbour class retains only product-projector hops, and the time reflection keeps  $PP^\pm$ . The selected member has one light Dirac field, with the free-field doubling and gauge-coupled positivity evidence set out below, and has a common leading kinetic cone under the stated symmetry assumptions. It is a Wilson-type action with  $O(a^2)$  anisotropy  $\kappa_f = +1/24$  exactly. Existing high-energy bounds give a

conditional spacing estimate if the free branch describes electrons and the stated source and emission assumptions apply (§8). [EXPLORATORY]

**Scope and reading guide.** The gravity sector studied here is a discretised Regge-type action placed on the tiling. It is not derived from the quantum register dynamics of the  $\mathbb{Z}^3 \otimes Q_3$  framework. The edge lengths are geometric variables of the tiling’s embedding; they are distinct from the framework’s matter registers. The paper establishes properties of a declared law. It does not claim that this law is selected by the framework, that it is the unique healthy choice, or that it describes nature. The fermion of §8 is likewise a declared construction on a declared four-dimensional premise, and that section is independent of the gravity sections. Sections 2–7 develop the gravity calculation; §8 can be read independently. The interpretation and priorities are collected in §9. The exact closure certificate and detailed numerical audit are in the appendices.

Throughout, a *ghost* in the gravity calculation means a physical mode with negative reduced kinetic energy; it is not a gauge-fixing ghost. A *doubler* is an additional unwanted light fermion produced by the lattice discretisation. The *Brillouin zone* is the finite range of inequivalent lattice momenta. A mode at the *cutoff* has frequency set by the inverse lattice spacing, rather than remaining light as that spacing is reduced. These are different tests: a sensible small-momentum dispersion does not by itself establish a positive physical norm or a consistent interaction.

## 2 The lattice and the law

### 2.1 The bond-centred bipyramid tiling

The matter cell of the  $\mathbb{Z}^3 \otimes Q_3$  framework is an oblate square bipyramid. In the bond-centred tiling, one bipyramid sits on each face shared by two cubes of a cubic arrangement. Its two apexes lie at the centres of the adjacent cubes, and its square equator lies on the shared face, so the apex-to-apex distance equals the equatorial edge  $a$ . The bipyramids tile  $\mathbb{R}^3$  with three face-normal orientations, and each has volume  $a^3/3$ . Each cell has 8 triangular faces and 12 edges: 8 lateral edges joining an apex to an equatorial vertex, and 4 equatorial edges.

The tiling has two vertex classes: cube centres (C, the apexes) and cube corners (K, the equatorial vertices). Per unit cell the lattice builder used below finds 11 edge classes, the 8 lateral and 3 equatorial directions, and 6 vertex-displacement gauge directions, three for each vertex class. Where a kinetic term needs a volume decomposition, each bipyramid is split into four tetrahedra around its apex–apex axis. The builder asserts that their volumes sum to the cell volume,  $1/3$  in units  $a = 1$ .

### 2.2 The linearised action

The gravitational variables are small changes in the edge lengths, not quantum register states. In Regge calculus an edge’s deficit angle measures the curvature concentrated there. The ADM decomposition separates spatial curvature from the kinetic energy of the spatial metric. We discretise these two pieces separately. The lapse fixes the local rate at which slices advance; the shift changes the spatial coordinate labels between slices. A *first-class* constraint preserves a gauge freedom rather than introducing a new physical mode.

The geometry sector is a classical linearised action for edge-length perturbations  $q(\mathbf{k})$  about the regular tiling, written in Bloch form at wavevector  $\mathbf{k}$ :

- a *Regge potential*, from the action  $S = \sum_E l_E \epsilon_E$  over tiling edges with deficit angles  $\epsilon_E$ , expanded to second order through the Schläfli identity. Its Hessian is  $H_R(\mathbf{k})$  and its coefficient  $C_R$ ;

- a *tetrahedral DeWitt kinetic term*  $\mathcal{M}(\mathbf{k})$  with weight  $C_R/2$ , evaluated on the four tetrahedra around each apex–apex axis (“axis-4”). On uniform strain it reduces exactly to the DeWitt supermetric [3], a guaranteed identity of the construction;
- *momentum (shift) constraints* generated by the gauge map  $G(\mathbf{k})$  of vertex displacements, with  $H_R G = 0$  (flat directions are pure gauge);
- *two vertex lapse constraints per unit cell*, one at each vertex class,

$$A(\mathbf{k}) = -C_R \begin{pmatrix} a \cdot \Phi_C(\mathbf{k}) \\ b \cdot \Phi_K(\mathbf{k}) \end{pmatrix} H_R(\mathbf{k}), \quad (1)$$

where  $\Phi_C$  and  $\Phi_K$  specify weighted sums over the local edge neighbourhood (the closed star) of each vertex. Edges related by the cubic point group  $O_h$  share a weight. The C star has orbits of 8, 12 and 24 edges (weights  $a_0, a_1, a_2$ , labelled C.L8, C.E12, C.L24). The K star has orbits of 8, 6, 24, 24 and 24 edges (weights  $b_0, \dots, b_4$ , labelled K.L8, K.E6, K.L24-near, K.L24-far, K.E24). The incident-lateral weights are fixed at  $a_0 = b_0 = \sqrt{3}/2$ .

There is no additional “rigid-displacement” constraint. The assembled action, coordinate conventions and exact coefficient table are specified in Appendix B; its stiffness is  $K = -C_R H_R$ .  $C_R$  cancels from every frequency ratio. The constrained system is reduced either by a Dirac route (constraints and their conjugate gauge directions removed) or by a configuration route ( $S = \ker[A; G^\dagger \mathcal{M}]$  with  $K_S v = \omega^2 \mathcal{M}_S v$ ); these are two gauges for the same reduction, not independent physical tests. The separate bordered-pencil check in §5 avoids that reduction. Here and in the gravity calculations  $a = 1$ , so  $\mathbf{k}$  is dimensionless. A physical momentum  $\mathbf{p}$  corresponds to  $\mathbf{k} = a\mathbf{p}$ ; the long-wavelength regime is  $|a\mathbf{p}| \ll 1$ .

**The exact-coefficient identification (premise E).** The snapshot of this action was built in floating point. Every one of the original 952 coefficients, including the predecessor’s vertex-lapse weights, was identified with a closed form  $(n/192)\{1, \sqrt{3}\}$ ,  $n$  an integer, with maximum deviation  $5.1 \times 10^{-13}$ . Under the identified law the flatness identity  $H_R G = 0$  holds to  $1.89 \times 10^{-50}$  at 50 digits, against  $8.3 \times 10^{-13}$  for the float law. The tiny residual is a high-precision check, not a proof that the floating-point construction determines unique exact coefficients. The exact results below concern the law defined by the identified coefficients. The identification itself is premise E; it is a numerical identification, not a derivation.

### 2.3 The lapse: maximal closure and GR normalisation

The lapse weights in (1) are not fixed by the construction. PL-1 uses the original Route 3 weights. The law studied here, PL-1N, makes two choices.

**Maximal closure (premise MC).** One lapse combination is required to close exactly, i.e. to be first class, wherever the lattice permits (§3). On this lattice that is the condition  $c = 0$  on the  $\langle 110 \rangle$  lines. One solution component is the linear subspace

$$L_1 : \quad a_1 = 0, \quad b_4 = 0, \quad b_2 + b_3 = \frac{\sqrt{3}}{2} - \frac{5\sqrt{3}}{9} b_1. \quad (2)$$

MC is a requirement placed on the law, comparable to  $H_R G = 0$ . It is neither a symmetry nor a dynamical protection, and it counts as one added input. Choosing  $L_1$  among the components of the  $c = 0$  set is a further disclosed choice; the other components contain stencils that fail bounded dynamics at lattice-scale  $|\mathbf{k}|$ .

**GR normalisation (premise N).** Within  $L_1$  the pure-gravity dynamics are the same along a two-parameter class (§5), so a representative must be chosen. It matters once matter couples at the vertex lapses. We choose the representative whose zero-momentum lapse pattern equals the continuum conformal pattern. This makes the static source coupling universal between C and K vertices and free of slip. The resulting weights are

$$R^* : \quad a = \left( \frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{18} \right), \quad b = \left( \frac{\sqrt{3}}{2}, 1, -\frac{\sqrt{3}}{24}, -\frac{\sqrt{3}}{72}, 0 \right). \quad (3)$$

These were first identified numerically, to  $3.6 \times 10^{-25}$ , with the representative constructed in 30-digit arithmetic. They are now taken as the definition of the lapse. They lie exactly on  $L_1$ , since  $-\frac{\sqrt{3}}{24} - \frac{\sqrt{3}}{72} = -\frac{\sqrt{3}}{18} = \frac{\sqrt{3}}{2} - \frac{5\sqrt{3}}{9}$ . An exact check confirms they are GR-normalised. GR normalisation counts as a further added input.

## 2.4 Inputs

Table 1 records the principal inputs. We distinguish the following kinds of evidence:

- [PROVED]: an exact argument, with an executed certificate in exact arithmetic;
- [SAMPLED]: evaluated at a stated finite set of points, with no general proof;
- [CONSISTENCY CHECK]: a property that holds by construction once the stated premises are imposed;
- [REVIEWED]: examined in a separate-context computational review, not external journal peer review (the limits of independence are described in Appendix D);
- [EXPLORATORY]: a provisional study whose assumptions and limited coverage remain part of the claim; its review and registration status are stated separately;
- [POST HOC]: a reading or computation decided after the registered result was seen.

| Input         | Content                                                                                                                                                                                                                                                     |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Geometry      | Bond-centred oblate square bipyramid tiling, regular reference configuration, units $a = 1$                                                                                                                                                                 |
| Action        | Regge potential $H_R$ ; axis-4 tetrahedral DeWitt kinetic term $\mathcal{M}$ ; shift constraints $G$ ; two vertex lapse rows (1); no rigid-displacement constraint                                                                                          |
| E             | Exact-coefficient identification $(n/192)\{1, \sqrt{3}\}$ of the snapshot                                                                                                                                                                                   |
| MC            | Maximal closure: lapse on $L_1$ (2) (one added input; choice of component disclosed)                                                                                                                                                                        |
| N             | GR normalisation of the representative, giving $R^*$ (3) (one added input)                                                                                                                                                                                  |
| P-N1, P-N2    | Static matter couples through the lapse multipliers as $N(H_{\text{grav}} + H_m)$ ; sign fixed by a continuum harness (§6 only)                                                                                                                             |
| P-T1, P-T2    | Slow test-particle couplings to lapse, metric and shift; $P_{\text{phys}} = 2P_{\text{code}}$ convention (§6 only)                                                                                                                                          |
| P-S           | Edge stress couples as $q^\dagger \Sigma$ , the lattice form of $\frac{1}{2}h_{ij}T^{ij}$ (§6.1 only)                                                                                                                                                       |
| P-ST1, P-ST2  | The four-dimensional bond-centred lattice with a uniform time step and point group $B_4$ ; a Euclidean action and the stated time-reflection construction (§8 only)                                                                                         |
| Fermion/gauge | A Dirac field on each four-dimensional cell; the PP+ projector hop; compact gauge links and the specified Wilson-type plaquette weights. The symmetric phase and a local long-wavelength description are assumed when interpreting the common kinetic cone. |

Table 1: Inputs. MC and N are choices made here, not consequences of the construction.

### 3 Constraint closure

The question is whether a constraint continues to remove a coordinate redundancy after the lattice replaces continuous space. Two statements must be kept separate. Exact closure holds at a fixed lattice momentum; long-wavelength closure holds to a specified order as the momentum tends to zero. PL-1N has the latter in every direction, but exact closure only on special lines within the region examined.

#### 3.1 Exact closure and the singular locus

On the lapse rows (1),  $AG = 0$  follows from  $H_R G = 0$ . Suppose  $A(\mathbf{k})$  has full row rank and  $\mathcal{M}(\mathbf{k})$  is invertible. Call a lapse combination  $w \neq 0$  *exactly closing* at  $\mathbf{k}$  if  $\mathcal{M}^{-1}A^\dagger w \in \text{range } G$ . In words, the kinetic image of that lapse constraint is a pure gauge direction, which is the lattice form of a first-class Hamiltonian constraint.

**Lemma 1** (exact closure). *Some lapse combination closes exactly at  $\mathbf{k}$  if and only if some  $\xi \in \ker G^\dagger \mathcal{M}G$  has  $\mathcal{M}G\xi \in \text{range } A^\dagger \setminus \{0\}$ . Exact closure therefore requires  $G^\dagger \mathcal{M}G$  to be singular. Where its kernel is one-dimensional, spanned by  $\xi_0$  with  $G\xi_0 \neq 0$ , exact closure holds if and only if  $c(\mathbf{k}) = 0$ . Here  $c$  is the component of  $p_0 = \mathcal{M}G\xi_0$  outside  $\text{span}\{V_C a, V_K b\}$ , with  $V_X = (\Phi_X H_R)^\dagger$ .*

The lemma is linear algebra; its proof and a 50-digit corroboration are in the source package. It was confirmed independently. [PROVED; REVIEWED]

**Proposition 2** (singular locus, sampled). *With the axis-4 kinetic term,  $G^\dagger \mathcal{M}G$  is positive definite at all 156,751 off-axis points of a grid of 156,846 points on the irreducible wedge (step  $\pi/96$ ,  $|\mathbf{k}| \geq 0.05$ ). It is singular on the  $\langle 110 \rangle$  lines, where it touches zero quadratically in the transverse angle.*

[SAMPLED; REVIEWED] Below  $|\mathbf{k}| \approx 0.05$  the smallest eigenvalue of  $G^\dagger \mathcal{M}G$ , relative to the largest, scales as  $k^4$  in every direction. This is the small-momentum approach to the continuum longitudinal gauge null. Additional reviewer samples covered smaller momenta without a counterexample; they do not prove the unsampled region.

On the  $\langle 110 \rangle$  lines the condition  $c = 0$  reduces to three exact bilinear equations in the stencil weights, with coefficients in  $\mathbb{Q}(\sqrt{3})$ . Equation (2) is one component of their solution set. [POST HOC] That reduction was found after the study's registered verdict, which was PARTIAL: the registered hypothesis of a single  $\mathbf{k}$ -independent condition failed, because the condition rotates with  $\mathbf{k}$  along the line. It is also a lattice remnant of a continuum identity: with DeWitt parameter  $\lambda = 1$ , the supermetric is null on longitudinal diffeomorphisms and dual to the Hamiltonian constraint. A reviewer found that this reading holds at small  $|\mathbf{k}|$  only; at lattice-scale  $|\mathbf{k}|$  it is a proposition.

#### 3.2 Closure to $O(k^2)$ in every direction

Write  $\mathbf{k} = h\mathbf{u}$  with  $|\mathbf{u}| = 1$ . The in-phase lapse row vanishes as  $h^2$ , so we use the normalised two-row matrix

$$\tilde{A}(h) = \begin{pmatrix} h^{-2}v(h)A(h) \\ e_C^T A(h) \end{pmatrix}, \quad v(h) = (1, 1) - h\lambda(\mathbf{u})e_C^T, \quad e_C^T = (1, 0). \quad (4)$$

Writing  $A(h) = \sum_j h^j A_j$ , the linear function  $\lambda$  is fixed by  $(1, 1)A_1 = \lambda e_C^T A_0$ . The second row retains the independent staggered lapse; omitting it would make the rank-two statement in the certificate impossible. The displacement columns are rescaled as

$$\tilde{G}(h) = [G(h)B_{\text{opt}}, h^{-1}G(h)B_{\text{ac}}], \quad B_{\text{opt}} = \begin{pmatrix} I_3 \\ 0 \end{pmatrix}, \quad B_{\text{ac}} = \begin{pmatrix} I_3 \\ I_3 \end{pmatrix}. \quad (5)$$

The column ordering is the C displacement followed by the K displacement. The identities  $(1,1)A_0 = 0$ ,  $(1,1)A_1 = \lambda e_C^T A_0$  and  $G_0 B_{ac} = 0$  make both quotients regular at  $h = 0$ . This normalisation is needed before comparing orders: the theorem concerns the residual in the equation below, not an exact finite-spacing constraint algebra or the full nonlinear diffeomorphism algebra.

**Definition 3.** A nonzero lapse combination *closes to*  $O(k^2)$  along  $\mathbf{u}$  if there are formal series  $w(h)$ ,  $\zeta(h)$  with  $w(0) \neq 0$  and  $\tilde{A}(h)^\dagger w(h) - \mathcal{M}(h\mathbf{u})\tilde{G}(h)\zeta(h) = O(h^2)$ .

This is equivalent to the vanishing of the first-order closure defect  $d_1$  used in the numerical studies. The equivalence uses:  $\mathcal{M}(\mathbf{k})$  is Hermitian, which holds exactly; rank conditions at  $h = 0$ , proved below for all directions; and the analytic-family argument stated in Appendix A. That argument is ordinary mathematics, not a machine-formalised proof.

**Theorem 4** ( $O(k^2)$  closure). *Under premise E, the lapse  $R^*$  of (3) admits a nonzero combination closing to  $O(k^2)$  along every direction  $\mathbf{u} \in \mathbb{R}^3 \setminus \{0\}$ . The same holds for the original Route 3 lapse used by PL-1.*

The certificate reduces the closure defect to polynomial identities in the direction  $\mathbf{u}$ . It proves that the relevant polynomials vanish, and separately checks the ranks needed to cover exceptional directions. Unlike an ordinary numerical scan, a finite exact interpolation test can establish an identity when a degree bound is known. Appendix A gives that bound, the certificate and the separate symbolic review.

Theorem 4 alone does not make a healthy law: PL-1 satisfies it and is unstable (§5.1). What distinguishes PL-1N is  $c = 0$  on  $\langle 110 \rangle$ , i.e. exact closure where exact closure is possible. The theorem says nothing about closure beyond  $O(k^2)$ . At fixed  $\mathbf{k}$  off  $\langle 110 \rangle$ , exact closure is impossible wherever  $G^\dagger \mathcal{M} G$  is non-singular (Lemma 1), which holds at every sampled off-axis point. That is a statement at each fixed  $\mathbf{k}$ ; whether closure holds to higher order in  $k$  was not examined.

## 4 Spectrum

Spectral quantities in this section were computed at nearcand0\*, the class member constructed first. By the exact transfer of §5 they are those of  $R^*$ ; a reviewer’s direct comparison agrees to  $10^{-24}$ . At each sampled  $\mathbf{k}$  the reduced system has 3 degrees of freedom: two massless modes and one massive mode.

**Massless modes.** Both massless modes are transverse-traceless, with TT fraction 1.0 to the printed precision. At long wavelength  $c^2 = 1$ , with  $|c^2 - 1| \leq 3.7 \times 10^{-15}$  on 26 rays by an infrared extraction. This agreement checks the implementation; it is not an independent prediction of the speed. The possible leading spin-2 anisotropy is the relative weight of the two cubic shear sectors, conventionally denoted  $E_g$  and  $T_{2g}$ , and the uniform-strain identity of the DeWitt term fixes it. So  $c^2 = 1$  and isotropy at  $O(k^2)$  are forced by construction. The tests beyond that identity concern the number and polarisation of the modes, the extra massive mode, and the higher-order dispersion and stability. Transverse-traceless means that the metric perturbation is perpendicular to the direction of propagation and has zero trace, as for the two gravitational-wave polarisations of linearised GR.

**Dispersion at  $O(k^4)$ .** Write  $\omega^2/k^2 = 1 + b k^2 + O(k^4)$  for each polarisation.

- On 25 of the 26 rays, both polarisations have  $b < 0$ , from  $-0.0277778$  to  $-0.00349$ .
- On [001], one polarisation has  $b = -0.0208333$ . The other has  $b = 0$  to the extraction precision, and its  $O(k^6)$  coefficient is  $-0.0041667$ .

- No sampled polarisation has a superluminal leading long-wavelength correction. This does not bound the front velocity or the dispersion at all momenta.

The two polarisations have different  $O(k^4)$  coefficients on the sampled rays, including [001]: the gravitons are birefringent at  $O(k^4)$ . Whether their  $O(k^4)$  coefficients coincide with those of the photon or of matter on the same lattice is not examined here.

**Massive mode.** One massive mode sits at  $\omega = 3.99992188$  for  $|\mathbf{k}| = 0.025$  and  $\omega = 3.999999875$  for  $|\mathbf{k}| = 10^{-3}$ , the same on every ray. Both values agree with  $4 - k^2/8$ . The mode lies at the cutoff scale.

The original registered spin-content test failed because its numerical rank threshold misclassified a near-dependency. A later test with the structurally determined rank passed on fresh rays. The original failure, the post-hoc correction and the control's limitations are retained in Appendix C.

On the exact  $\langle 110 \rangle$  lines, after quotienting the structural null, there are again two light modes and one massive mode. At  $|\mathbf{k}| = 0.3$ ,  $\omega^2 = 0.089829622$ ,  $0.089830808$  and  $15.910588$ .

## 5 Stability across the Brillouin zone

### 5.1 The failure being repaired

PL-1 has the same action with the original Route 3 lapse, which closes to  $O(k^2)$  (Theorem 4) but has  $c \neq 0$  on  $\langle 110 \rangle$ . Its reduced dynamics has a kinetic ghost: the reduced kinetic matrix has a negative eigenvalue while the potential stays positive, and the mode grows exponentially.

- **Where.** On a 3080-point zone scan, growth appears at 6 of 3071 resolved points, all close to a  $\langle 110 \rangle$  axis.
- **Shape of the region.** A 50-digit probe of the identified law maps it as a thin wedge around each axis, with an elliptical cross-section. The onset ratio  $k_{\min}/\theta$  is 39.2–39.4 towards  $[1\bar{1}0]$  and 31.0 towards  $[001]$ . These are the independent review's exact values; the original grid gave upper bounds.
- **Reach.** The wedge reaches the probe floor  $|\mathbf{k}| = 0.02$ , where the growth rates are  $1.45 \times 10^{-2}$  and  $1.05 \times 10^{-2}$ .
- **Review.** An independent review confirmed the instability with qualifications. The ghost is the TT “+” polarisation, and it disappears if the near-first-class acoustic lapse is classified as first class.

Maximal closure makes that classification exact on  $\langle 110 \rangle$ .

### 5.2 PL-1N over the zone

**Exact transfer between representatives.** On the pure-gravity sector the lapse weights enter only through the constraint subspace that the rows (1) define. Three exact Laurent-polynomial identities in  $\mathbb{Q}(\sqrt{3})$  show that this subspace is the same for every member of the class  $\{L_1, b_2 = \frac{\sqrt{3}}{3} - \frac{3\sqrt{3}}{8}b_1, b_1 \neq 0, |a_2| < \frac{\sqrt{3}}{6}\}$  at every real  $\mathbf{k}$ :

$$\rho_{C2} \equiv \left( \sum_i \cos k_i \right) \rho_{C0}, \quad (6)$$

$$3\rho_{K0} + 2\rho_{K2} + \rho_{K3} \parallel \rho_{C0}, \quad (7)$$

$$|\rho_{C0}|^2 \equiv 72 + 12 \sum_{i < j} (1 + \cos k_i)(1 + \cos k_j), \quad (8)$$

where  $\rho_{Xj}$  are the single-orbit rows. The last identity gives  $|\rho_{C0}|^2 \geq 72$ .  $R^*$  lies exactly in the class.

- **What carries over.** Linear constrained dynamics at fixed  $\mathbf{k}$  depends on the constraint rows only through their span. Pure-gravity spectra and stability therefore carry over exactly between class members at every  $\mathbf{k}$ . A reviewer confirmed this numerically to  $10^{-24}$  for the reduction routes used here.
- **What does not.** The change of row basis is local in  $\mathbf{k}$  but non-local in space. Couplings placed at the vertex lapses therefore see the representative (§6).

[IDENTITIES PROVED; REVIEWED]

**Numerical evidence.** Direct samples at  $R^*$  found no growing physical mode or negative reduced energy. The samples include generic zone points, narrow angular approaches to the face diagonals, zone corners and surfaces where the unreduced kinetic matrix becomes singular. Agreement between the two gauges was checked against a separate bordered-pencil inertia calculation. High-precision controls detect the known instability when exact closure is spoiled. The sample sizes, exceptional points and original undecided verdict are given in Appendix C. [SAMPLED; REVIEWED]

**What is not shown.** Stability is sampled, not proved. In particular:

- $\mathcal{M}$  is singular on two-dimensional surfaces and at the zone corner. The dynamics stay positive there at the sampled points.
- No complete proof covers the structural-null lines.

**Fragility.** Ghost-freedom near  $\langle 110 \rangle$  needs  $c = 0$  exactly.

- **Rounding.** Any double-precision implementation of  $R^*$  has  $c \approx 10^{-16}$  again, which opens a ghost wedge below about  $10^{-18}$  rad. This is a failure to preserve the defined exact relation in numerical arithmetic. It is distinct from the unanswered physical question of whether interactions preserve that relation.
- **Finite tolerance of  $O(k^2)$  closure.** Stability also needs  $O(k^2)$  closure to a finite tolerance. Moving  $b_1$  or  $b_2$  by 0.1 while staying on  $L_1$  produces ghosts or growth at 343 and 658 of 3080 zone points.
- **No protection.** No symmetry protecting  $c = 0$  is known.

## 6 Source couplings: consistency checks

Recovering a gravitational-wave spectrum does not establish how matter gravitates. Here the source and test-particle rules are additional assumptions. They test the compatibility of PL-1N with familiar weak-field responses, rather than deriving a universal stress tensor from a joint matter-gravity action. A static vertex energy enters through the lapse multipliers as  $N(H_{\text{grav}} + H_m)$  (P-N1), with the sign fixed by a continuum harness (P-N2). Slow test particles couple to lapse, metric and shift (P-T1, P-T2).

**Static limit.** Under these premises, both the PL-1N law (at  $R^*$ ) and the comparison law of §7 give a static response with these properties:

- an attractive, isotropic  $1/k^2$  pole;
- no gravitational slip ( $\eta = 1$ ): the two scalar metric potentials governing slow motion and spatial curvature agree;

- no static singularity at the 3080 zone points, nor at 300 further points sampled by a reviewer;
- the same pole coefficient,  $c_1 = -2.0$  in the normalisation where the continuum harness gives  $c_1 = -4$  at  $16\pi G = 1$ . At this lattice normalisation that is the same Newton constant as the kinetic term's.

These are consistency checks, not predictions. Across lapse stencils the pole coefficient follows  $c_1 = -4\eta_K^2/(1+r^2)$ , where  $r$  is the C-to-K response ratio and  $\eta_K$  the slip. This was verified on 7 package stencils and on 8 of 8 fresh GR-normalised stencils; it is not derived. A non-normalised control gives  $c_1 = -2.1900452$ , as the formula predicts.

- GR normalisation sets  $r = \eta_K = 1$ , which forces  $c_1 = -2$ .
- The pole's isotropy is guaranteed by  $O_h$ .
- Its existence and coefficient follow from the Regge conformal-mode stiffness.

Before normalisation, the representative constructed earlier gives  $r = 1.26848623807$  and  $c_1 = -1.53004522606$ . This is the representative dependence noted in §5.

**Stationary gravitomagnetism.** This is the weak-field response to a steady mass current, analogous to the magnetic response to an electric current. For a stationary transverse shift sourced by matter momentum, the response coefficient is  $c_T = 0.125$  ( $= C_R/8$  in code units, the DeWitt value of GR at  $O(k^2)$ ). This holds on 26 rays and both polarisations, for both laws. The target value was fixed after a convention correction made post exposure:  $P_{\text{phys}} = 2P_{\text{code}}$ , part of premise P-T2. [CONSISTENCY CHECK; REVIEWED]

**Not linearised GR.** The longitudinal shift kernel is  $c_{L4}k^4$ , and  $c_{L4}$  is not zero. Over the sampled rays it ranges up to 0.00520833 for PL-1N (with  $c_{L4} \approx 0$  on [110]), and from 0.00104167 to 0.00219907 for the comparison law. GR has exactly zero. The non-stationary and conservation sector therefore differs structurally from GR; §6.1 tests it. For PL-1N, exact closure plus the normalisation constrains energy continuity only on the  $\langle 110 \rangle$  Fourier lines, and that statement depends on the representative.

## 6.1 Longitudinal sector

[EXPLORATORY: CONSTRUCTOR-CHECKED, NOT REVIEWED] A dedicated test addressed the questions this kernel raises, for both laws, under one further premise: edge stress couples to the metric perturbation as  $q^\dagger \Sigma$ , the lattice form of ADM's  $\frac{1}{2}h_{ij}T^{ij}$  (P-S).

- **Run record.** The first registered run was void because one harness row was registered against the wrong combination of vertex momenta. A dated addendum corrected that row before the second run, so the second run's outcomes are post-exposure.
- **Verdict.** The registered verdict of the second run is LATTICE-SUPPRESSED for both laws.
- **Code.** `technical_notes/wp3_longitudinal_shift_2026-10-16/`.

**Gauge status.**  $c_{L4}$  is the longitudinal eigenvalue of  $G^\dagger \mathcal{M} G$ . It depends only on  $\mathcal{M}$  and  $G$ , so it is the same for every lapse representative. At the generic sampled point the null space of the full linear system has dimension 6: the spatial diffeomorphisms alone. Away from the poles, the system’s smallest singular value relative to its largest is non-zero and scales as  $|\mathbf{k}|^{4.001}$  on every informative ray for both laws; in the continuum it is exactly zero. So  $c_{L4}$  is physical, not gauge. It measures the lattice’s explicit breaking of time reparametrisation.

On  $\langle 110 \rangle$  the picture differs for PL-1N. An independent reviewer found 7 null vectors there, the six spatial diffeomorphisms plus time reparametrisation, with the seventh singular value at most  $5.8 \times 10^{-52}$  of the largest. This holds both at the closed-form  $R^*$  and at the stored weights. PL-1R and PL-1 have only the six. Maximal closure therefore restores an exact time-reparametrisation symmetry on those lines. [REVIEWED]  $c_{L4} = 0$  is not enough by itself: PL-1 has  $c_{L4} = 0$  on  $[110]$  but no seventh null.

### Continuity.

- **Momentum.** Both laws force exact lattice momentum conservation,  $i\Omega P = G^\dagger \Sigma$ , at every  $\mathbf{k}$ . A source whose momentum is scaled by 1.1 is inconsistent (residual 0.011).
- **Energy.** The field equations do not force energy continuity off  $\langle 110 \rangle$  for PL-1N, nor anywhere for PL-1R. Conserved sources are consistent at every order tested, and so are non-conserved ones.
- **Response to non-conserved sources** [PROPOSITION]. The observed conditioning permits a potentially enhanced response to sources overlapping the near-null channel. Its magnitude and scaling for a specified continuity-violating source have not been calculated; the relative singular value alone does not determine them.

**A spherical breathing-source diagnostic.** In continuum GR, Birkhoff’s theorem forbids a changing spherically symmetric vacuum exterior of this kind. A lattice source test is more limited: cubic symmetry and the map from continuum stress to edges can themselves introduce anisotropy. At all 52 sampled points, both laws have exactly two light modes, the TT pair with positive residue norms, and one massive mode at  $\omega \geq 3.995$ . No additional light scalar or longitudinal mode, or negative physical residue, was found at these points.

A breathing, mass-conserving source nevertheless excites the TT pair. Its squared mode coupling, normalised to that of a TT quadrupole with equal stress amplitude, scales as  $|\mathbf{k}|^n$ :

- PL-1N:  $n \in [3.9993, 4.0007]$ ;
- PL-1R:  $n \in [3.9967, 4.0013]$ .

These ranges come from 36 fits: 9 informative ray–law cases times 4 source variants. At  $|\mathbf{k}| = 0.025$  the ratio is at most  $4.4 \times 10^{-11}$  (PL-1N) and  $1.8 \times 10^{-10}$  (PL-1R). The PL-1N computation used the stored 30-digit lapse weights, which differ from the closed-form  $R^*$  by  $3.6 \times 10^{-25}$ . For this source prescription the per-mode amplitude is therefore suppressed as  $(ka)^2$ , and its square as  $(ka)^4$ . These are not results for the total power radiated by a finite body. The massive mode, the only scalar-like channel, can be emitted only at the cutoff.

**Reading.** The sampled linear calculation found none of the targeted failures: a light extra mode, a ghost, or an unsuppressed monopole leak. The test does not discriminate between the two laws.

**Strongest objection.** [PROPOSITION] The  $(ka)^4$  leak may reflect the map from a continuum source onto the lattice and the lattice’s cubic anisotropy, rather than  $c_{L4}$ . Two observations support this:

- two different stress maps give the same exponent;
- both laws agree, although their  $c_{L4}$  differ ray by ray, and PL-1N’s vanishes on [110] while PL-1R’s vanishes on no sampled ray.

On the light shell,  $k = \Omega/c$ , the same per-mode squared-coupling estimate is proportional to  $(\Omega a/c)^4$ . The tested residues are positive. Its origin and integrated physical significance remain open.

**Not addressed.** Nonlinear Birkhoff; the integrated power radiated by a finite source (only per-Fourier-mode couplings were computed); and the continuity law of real lattice matter.

## 7 Dependence on the discretisation

PL-1R keeps PL-1’s Route 3 lapse and replaces the axis-4 kinetic decomposition by eight tetrahedra joining the cell centroid to each face (“cone-8”). This respects each cell’s  $D_{4h}$  symmetry and involves no axis choice. Both kinetic decompositions give the exact DeWitt form on uniform strain, so the  $O(k^2)$  graviton speed is unchanged.

[SAMPLED; EXPLORATORY] Under cone-8,  $G^\dagger \mathcal{M} G$  is positive definite at every sampled  $\mathbf{k} \neq 0$ , so maximal closure is vacuous: no lapse closes exactly anywhere. PL-1R passes PL-1’s zone tests. Its kinetic margin on [110] comes from kinetic energy that cone-8 assigns to the centroid’s motion, an interior reparametrisation of the cell.

The two laws differ at  $O(k^4)$ . PL-1R has  $b > 0$  for one polarisation on 25 of 26 rays, i.e. a superluminal  $O(k^4)$  correction, where PL-1N has none. They agree on every source coupling tested in §6 at leading order. They differ in the lattice-scale longitudinal kernel, but not in the scaling exponent of the breathing-source diagnostic in §6.1. This comparison matters because the leading continuum checks do not select a unique discretisation: different finite-spacing theories can share them.

## 8 Fermions on the four-dimensional extension

The gravity calculation leaves the matter dynamics unspecified. We now consider an explicit Dirac action on the four-dimensional extension of the same cell construction. A lattice Dirac operator must pass two different tests. It should avoid extra light copies of the intended fermion, and its Euclidean correlations should admit a positive physical inner product. The latter is the purpose of *reflection positivity*: correlations between observables and their time-reflected partners must define non-negative squared norms after null states are removed. A transfer matrix then relates adjacent time slices. The choice of reflection plane matters here.

The construction is exploratory. Its four source packages were registered before their respective calculations, with amendments and post-hoc steps disclosed below. Separate-context computational reviews replayed the packages and checked the mathematics and literature; their conditions are incorporated in this draft. These were internal computational reviews, not external peer review. The later Wilson comparison and scale estimates were unregistered checks. Appendix D records the sources and review limits.

## 8.1 Premises, class and member

**P-ST1 and P-ST2.** The tiling is the exact centre-time slice of the bond-centred decoration of  $\mathbb{Z}^4$ : one four-dimensional cell on every bond of  $\mathbb{Z}^4$ , with the time step equal to the spatial step. The point group is the hypercubic group  $B_4$ , which treats Euclidean time like a spatial axis (P-ST1). The dynamics is specified by a Euclidean action, with a transfer-matrix interpretation sought through the reflection construction below (P-ST2). Equal temporal and spatial steps are supplied, not derived clocks or rulers. Both are declared premises. Neither is derived from the framework, and neither changes the three-dimensional gravity sections.

**The class  $C'$ .** One Euclidean Dirac 4-spinor sits on every cell. Couplings are on-site and between cells that share a facet, which are the cells of two orthogonal bonds of  $\mathbb{Z}^4$  meeting at a vertex; each cell has twelve such neighbours. For such a pair, let  $u$  be the unit vector from the first cell to the shared vertex and  $v$  the unit vector from that vertex to the second cell, with  $d = (u + v)/2$  and  $a_b = (u - v)/2$ . Here  $a_b$  is a bond vector, distinct from the lattice spacing  $a$ . The stabiliser of an oriented pair in  $B_4$  has 8 elements and leaves invariant a 4-dimensional space of  $4 \times 4$  matrices, spanned by  $\mathbb{K}$ ,  $\gamma \cdot d$ ,  $i\gamma \cdot a_b$  and  $(\gamma \cdot u)(\gamma \cdot v)$ ; the cell stabiliser has 96 elements and fixes only  $\mathbb{K}$ . With  $\gamma_5$ -hermiticity,  $\gamma_5 D \gamma_5 = D^\dagger$ , the coefficients are real (real nullity 5 of the ten real parameters), so the bond operator is

$$V_b = -r\mathbb{K} + \kappa \gamma \cdot d + iw \gamma \cdot a_b + \delta (\gamma \cdot u)(\gamma \cdot v), \quad D_{cc} = m + 12r, \quad (9)$$

where the Euclidean gamma matrices obey  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ , and  $\kappa = 2/3$  is fixed by the discrete-exterior-calculus normalisation and  $(r, w, \delta)$  free. The on-site convention makes the symmetric mode's mass exactly  $m$  at  $k = 0$  for every member.

**The member PP+.** At  $(r, w, \delta) = (\kappa/2, 0, -\kappa/2)$  each hop is proportional to a product of two spinor projectors. Each factor selects the spinor components associated with one leg of the hop:

$$V_b = -\frac{\kappa}{2} (1 - \gamma \cdot u)(1 - \gamma \cdot v), \quad (10)$$

and in Bloch form, with the phase  $e^{ik \cdot (c_2 - c_1)}$  between cell centres, the row block on the first cell and  $U_\mu(k) = e^{-i\gamma_\mu k_\mu/2}$  the Wilson  $r = 1$  half-step hop,

$$D_{\mu\nu}(k) = M \delta_{\mu\nu} - 2\kappa (1 - \delta_{\mu\nu}) U_\mu U_\nu, \quad M = m + 6\kappa, \quad (11)$$

which agrees with the operator built on the lattice to  $9.2 \times 10^{-16}$  and was re-derived from the hop rule by the review. The product order is physical, not conventional: PP−, with both signs reversed, is the other product in this order, and the two reversed-order products  $-\frac{\kappa}{2}(1 - \gamma \cdot v)(1 - \gamma \cdot u)$  and its sign reversal, at  $(r, w, \delta) = (\pm\kappa/2, 0, \pm\kappa/2)$ , complete the four product-projector members of  $C'$ .

## 8.2 Properties of the free field

- **Covariance and speed.**  $D$  is covariant under all 768 lifts of  $B_4$  to the double cover on the  $L = 3$  torus; reversing the projector order on the time rows alone drops this to 192. The leading massless kinetic coefficient gives  $c = 1$  by  $B_4$ , in units of the equal temporal and spatial steps:  $|c^2 - 1| \leq 1.1 \times 10^{-12}$ . [CONSISTENCY CHECK]
- **One light Dirac field, no doublers.** At  $m = 0$ ,  $\dim \ker D(0) = 4$ ; on a  $16^4$  momentum grid at  $m = 0.05$  the smallest singular value off  $\Gamma$  is 0.4004, well above  $3m$ , and the review found  $k = 0$  to be the only singular zone corner. The massless real-momentum statement also follows directly from (14): every  $d_\mu$  is positive, and the remaining factor is a sum of real squares. A zero requires  $\sin k_\mu = 0$  for every  $\mu$ ; at a corner with  $n_\pi$  components equal

to  $\pi$ ,  $S = 3n_\pi/4$ . Thus only the origin is a massless zero. This argument is for the free operator, not an interacting phase diagram. The twelve other components of  $D(0)$  are algebraically heavy; their number is not a count of additional propagating particles. The transfer-root analysis finds a single spin-degenerate forward branch, with its backward partner. On a post-hoc  $4^3$  spatial grid at  $m = 0.1$ , that branch is positive near the light pole and negative at some large momenta. These negative values are a continuation of the branch, not twelve extra heavy particle species; their scale is quoted below. [POST HOC]

- **Reflection positivity.** For the time reflection  $\theta_T$  through a plane of time cells, the Osterwalder–Schrader Gram matrices are positive in both sectors at all 80 spatial momenta tested, with worst eigenvalue ratio  $-4.2 \times 10^{-15}$ , consistent with zero at numerical precision. The mechanism is Wilson’s at  $r = 1$  [12, 13]: every bond operator from a time cell to the space cell above it satisfies  $(1 + \gamma_4)V_b = 0$  exactly, on all 6 such bonds, so the cross-plane term factorises in the Osterwalder–Seiler form. Within  $C'$ , PP+ is the unique member with this property and PP− the unique member with  $(1 - \gamma_4)V_b = 0$ ; the reversed-order products satisfy the right-projector identity  $V_b(1 \pm \gamma_4) = 0$  instead.
- **Selection within the stated class.** Of 450 grid members of  $C'$  and 52 neighbours of PP± at offsets of  $\pm 0.02$ , exactly 4 survive the screen, the four product-projector members in either order, and none of the neighbours. In the full test the two reversed-order products pass the site reflection but fail  $\theta_T$  at 79 of the 80 momenta and carry an exact doubler at the zone corner  $k = (\pi, \pi, \pi, \pi)$ , where the smallest singular value off  $\Gamma$  equals the mass; PP± remain. The exact left-projector criterion selects PP± within  $C'$ . The finite parameter screen alone is not a classification of every reflection-positive action, even within a larger nearby class. Choosing the class and this criterion remains an action-level assumption.
- **Cost.** The site reflection  $\theta_S$  fails at 53 of the 80 momenta; exactly there the one-step transfer matrix has negative eigenvalues, with  $|z| \leq 0.1922$ , i.e.  $E \geq 1.649$  in lattice units. For a real transfer eigenvalue  $z$ , the two-step energy is  $E = -\log |z|$  in one-step time units: a negative  $z$  adds an alternating sign between adjacent slices. The Hamiltonian is obtained from  $T^2$  on its non-null support. Thus time-cell reflection positivity does not imply positivity of the one-step transfer matrix or of the other reflection. The light-mode ratio agrees on four sampled rays:  $(E^2 - m_{\text{pole}}^2)/k^2 = 1.000757$  on four rays at  $m = 0.1$ ; the offset scales as  $m^2$  in the review’s check and is expected to be an  $O(m^2 a^2)$  lattice effect, which was not isolated analytically. [POST HOC]

### 8.3 Gauge coupling and the common leading cone

The fixed-geometry matter–gauge action, plaquette weights and reflection conventions are specified in Appendix B. The group  $B_4$  consists of signed permutations of the four Euclidean axes. At quadratic order its scalar momentum invariant is  $p_4^2 + \mathbf{p}^2$ . For an isolated light mode with a local analytic inverse propagator, an unbroken  $B_4$ -symmetric action, measure and state therefore give the same coefficient to temporal and spatial kinetic terms. Species may have different overall normalisations; these can be removed by field rescaling. They cannot have independent leading temporal-to-spatial ratios under these assumptions. Reflection positivity is additionally needed to interpret the Euclidean coefficient as a physical propagation speed. This argument concerns the leading continuum kinetic terms. It does not imply exact Lorentz symmetry at finite spacing, equal higher-order dispersion, or a symmetry-preserving physical vacuum for the gravity sector.

- **Coupled action.** A bridge is a link between neighbouring cell centres; gauge parallel transport is placed on it. With U(1) or SU(3) link variables on these bridges, the action is  $B_4$ -covariant for 768 of 768 lifts and gauge invariant; 384 of 384 group elements leave

the Wilson-type plaquette action on the bridge complex invariant, against 96 for a control with one plaquette class reweighted.

- **Positivity on backgrounds.** The finite test used  $m = 0.1$  on a spatial-period-three, time-period-six torus, with periodic spatial and antiperiodic fermion time boundaries; the sign twist is placed at the far fixed plane. The earlier periodic-time prescription failed even for the free control. The compatible oriented-background rule is  $U_{\theta_T a \rightarrow \theta_T b} = U_{a \rightarrow b}$ , with adjoints on reversed links. Under these conditions,  $\theta_T$  positivity holds in both sectors on all 14 reflection-invariant backgrounds tested, 7 U(1) and 7 SU(3) (unit, Haar-random and near-identity), with worst ratio  $-2.4 \times 10^{-15}$ . The O2a member with  $r = 0.25$  fails, and, in a reported-only row of the registration, a  $\delta$ -shifted PP+ fails at  $-3.5 \times 10^{-3}$ , as predicted from the free field. The integrated-measure argument uses the exact cross-plane fermion factorisation together with a non-negative character expansion of the reflection-compatible Wilson-type plaquette action. This invokes the Osterwalder–Seiler construction [13]; it is not inferred from the finite background sample. Appendix B states the reflection geometry, boundary determinant,  $M > 0$  condition and non-negative plaquette-weight hypotheses. The saved evidence does not provide a separate complete proof of the integrated construction for all volumes and parameters; the extension remains a conditional proposition. It concerns prescribed geometry, not a dynamical matter–gravity action. [PROPOSITION]
- **One loop.** The compact-U(1) calculation uses  $e^2 = 1$ , Feynman-type gauge, a finite symmetric  $N = 12$  momentum grid, and a  $B_4$ -invariant photon-mass regulator  $\mu^2 = 0.01$  in lattice units. Charged-field coefficients are extracted at  $h = 0.02$  and  $0.04$ . No regulator-removal or infinite-volume limit is established; these are Euclidean symmetry checks, not renormalised physical-pole dispersions. The charged species’ ratio along the two  $B_4$ -related axes satisfies  $|c_e^2 - 1| = 3.6 \times 10^{-12}$  at  $m = 0.3$  and  $2.1 \times 10^{-12}$  at  $m = 0.6$ ; the photon’s six acoustic eigenvalues along the same axes agree to  $1.3 \times 10^{-13}$ ;  $c_e^2 - c_\gamma^2 = 3.7 \times 10^{-12}$ . The equality of the Euclidean  $O(k^2)$  coefficients is guaranteed by  $B_4$  invariance of the action, the measure and the momentum grid; positivity is additionally required for their interpretation as speeds. The reported axis equality is not a complete calculation of the dispersion at a finite physical pole mass. [CONSISTENCY CHECK] The axis criterion and the control below are the study’s third amendment, adopted after the registered all-direction criterion had failed: the one-loop coefficients on the diagonal directions differ from the axes by  $1.5 \times 10^{-6}$  at  $m = 0.3$  and  $3.9 \times 10^{-8}$  at  $m = 0.6$ , against  $2.5 \times 10^{-9}$  at tree level, a difference that could arise from the higher-order terms permitted at finite momentum. A controlled momentum-extrapolation or error bound has not established that attribution; it remains a hypothesis. A useful sensitivity check is supplied by the control: a plaquette reweighting that breaks  $B_4$ , compensated so that the tree-level acoustic tensor is isotropic, gives the charged species a one-loop anisotropy between the two axes of 0.0318 (0.0207 at  $\alpha = 2$ ), so the instrument can see a one-loop split; the registered control, which compensated with a different plaquette class, had converged to a  $B_4$ -symmetric point and could not split. The gluon’s non-Abelian loops were not computed; only the conditional leading-coefficient symmetry argument applies to that sector.

## 8.4 Relation to Wilson fermions and the leading lattice correction

**Map to Wilson fermions.** From (11),

$$D_{\text{PP}+} = \left[ M - 2\kappa \sum_{\mu, \nu} |\mu\rangle U_\mu U_\nu \langle \nu| \right] + 2\kappa \text{diag}_\mu(U_\mu^2). \quad (12)$$

A Schur complement is the effective quadratic operator left after auxiliary fields have been integrated out. The bracket is that operator, restricted to the link cells, of a Wilson  $r = 1$

half-step fermion living on the vertices and links of  $\mathbb{Z}^4$ : with vertex fields  $\psi_x$  of mass  $M_v$ , link fields  $\chi_l$  of mass  $M$  and vertex-link hops  $-\lambda [\bar{\chi}_l P_n \psi_x + \bar{\psi}_x P_{n'} \chi_l]$  through the half-step projectors  $P_n = \frac{1}{2}(1 - \gamma \cdot n)$ , integrating out  $\psi$  gives the link-link hops  $-(\lambda^2/M_v)P_u P_v = V_b$  with  $\lambda^2/M_v = 2\kappa$ , together with a collinear full-step hop  $U_\mu^2$  between the two links on either side of a vertex, which the second term removes; the back-hop to the same link vanishes because  $P_u P_{-u} = 0$ . The identity holds at random complex  $k$  to  $8.9 \times 10^{-16}$ , and the review took the Schur complement of the explicit vertex-link action. PP+ has no chiral symmetry ( $\{\gamma_5, D\} \neq 0$  at  $m = 0$ ) and its mass renormalises additively, so it is in Wilson's class [11, 14]. The proposed distinction is the link-midpoint site set and the ordered two-projector hop. The limited literature search described in Appendix D does not establish priority or uniqueness.

**The  $O(a^2)$  anisotropy.** Equal leading speeds still leave observable differences at higher momentum. For a massless species with a single non-birefringent light branch,  $B_4$  restricts the leading scalar dispersion correction to

$$E^2 = k^2 + \kappa_s a^2 \left( k^4 + \sum_i k_i^4 \right) + O(a^4), \quad (13)$$

where  $E$  and  $\mathbf{k}$  in this equation are physical frequency and momentum after restoring  $a$ ,  $k^2 = |\mathbf{k}|^2$ ,  $k^4 = (|\mathbf{k}|^2)^2$ , and the remainder is an expansion at fixed physical momentum. The quartic  $B_4$  invariants are spanned by  $(k^2)^2$  and  $\sum_\mu k_\mu^4$ , and the  $O(4)$  invariant drops out at the displayed on-shell order. This scalar argument alone does not exclude additional polarisation structures for an arbitrary gauge action; the photon statement below uses the computed physical roots. For PP+ the coefficient is exact. The determinant of (11) factorises as

$$\det D(k) = \prod_\mu d_\mu^2 \left( S^2 + 4\kappa^2 \sum_\mu \beta_\mu^2 \right)^2, \quad (14)$$

$$d_\mu = M^2 + 4\kappa^2 + 4\kappa M \cos k_\mu, \quad \beta_\mu = \frac{M \sin k_\mu}{d_\mu}, \quad S = 1 - 2\kappa \sum_\mu \frac{M \cos k_\mu + 2\kappa}{d_\mu},$$

and at  $m = 0$  the scalar factor expands as  $Zk^2 + A(k^2)^2 + B \sum_\mu k_\mu^4$  with  $Z = 9/256$ ,  $A = 9/4096$  and  $B = 3/2048$ , so that  $\kappa_f = B/Z = 1/24$  exactly, the  $A$  term entering on shell only at  $O(k^8)$ . The numerical extractions agree:  $\kappa_f = 0.0416666679$  from the lattice spectra, with identity spread  $2.1 \times 10^{-8}$  over six rays, and  $0.0416667$  on an axis and  $0.0416667$  on  $[111]$  from the closed form (Richardson at  $|k| = 0.04$  and  $0.08$ ), against  $-0.333332$  and  $-0.333332$  for Wilson  $r = 1$  and the same  $-1/3$  for naive fermions, whose value follows from  $\sin^2 q = q^2 - q^4/3 + O(q^6)$ . The photon on the same lattice has  $\kappa_\gamma = -0.0416666654$ , from a recount of the near-cone roots that removes a ghost-site root included in the registered count, whose value is superseded; numerically  $\kappa_\gamma + 1/24 = 1.3 \times 10^{-9}$  and  $\kappa_f + \kappa_\gamma = 2.5 \times 10^{-9}$ . The review reproduced  $\kappa_\gamma$  with an independent builder of the gauge complex to  $1.8 \times 10^{-8}$ , derived the ghost branch's coefficient  $-1/48$  exactly, and found that a  $B_4$ -preserving reweighting of the square plaquettes moves  $\kappa_\gamma$  to  $-0.0333$  while keeping  $c = 1$ : the photon's value is fixed by the discrete-exterior-calculus weights, not by symmetry, and  $\kappa_\gamma = -\kappa_f$  is not derived. In each action's own lattice units PP+ therefore carries an eightfold smaller  $O(a^2)$  anisotropy of the opposite sign, fourfold at equal spinor density since PP+ has four spinors per site of  $\mathbb{Z}^4$ : a partially improved Wilson-type action, aimed at reducing free-dispersion errors, as are Brillouin fermions [18]. On the body-centred hypercubic lattice the larger point group  $F_4$  admits no quartic invariant beyond  $(k^2)^2$ , as checked by the invariant calculation listed in Appendix D. This removes that scalar anisotropy. Celmaster's gauge action uses the enlarged symmetry [16]; Celmaster and Krausz found that naive fermions on that lattice have doublers that do not propagate as particles, leaving the continuum theory non-Lorentz-invariant [17]. For the stated free actions the leading group-velocity corrections are

superluminal for the massless fermion and subluminal for the photon. This statement concerns the light branches in the long-wavelength expansion, not a bound on signal fronts.

## 8.5 Conditional bounds on a physical lattice spacing

[PROPOSITION; ORDER OF MAGNITUDE] In lattice field theory the terms in (13) are irrelevant operators removed by  $a \rightarrow 0$  [15]. If the lattice is instead physical with finite  $a$ , the fixed action predicts residual dispersion, and  $\kappa_f > 0$  allows vacuum Cherenkov emission by electrons above  $p_{\text{th}} = (m_e^2/3ca^2)^{1/4}$ , with  $c$  here denoting a dispersion coefficient, not the speed of light:  $c = 2\kappa_f$  on an axis and  $4\kappa_f/3$  on [111] [21]; the subluminal photon does not lower this soft-photon threshold at leading order. The observational argument assumes a nonzero electromagnetic emission amplitude and sufficiently rapid energy loss above threshold, as in the effective-field-theory treatment. A PP+ emission rate has not been calculated. Under that assumption the relevant datum is the existence of high-energy electrons: electrons of about 2.3 PeV, inferred under the leptonic interpretation that LHAASO adopts from the 1.1 PeV photon of the Crab nebula [22], where a hadronic contribution is not excluded, require  $p_{\text{th}}$  above their energy. This is the bound of Li and Ma [23],  $E_{\text{LV},2} > \sqrt{3} E_e^2/m_e$ , re-expressed in the lattice spacing. Since the Crab’s direction in the lattice frame is unknown, the direction-independent bound is the [111] value  $a < 4.7 \times 10^{-32}$  m, about  $2.9 \times 10^3$  Planck lengths; the strongest case, on an axis, is  $a < 3.9 \times 10^{-32}$  m, or  $2.4 \times 10^3$  Planck lengths. These are conditional exclusions, not evidence for a nonzero lattice spacing.

The photon’s quadratic dispersion limit  $E_{\text{QG},2} > 1.3 \times 10^{11}$  GeV, the total-dispersion limit from GRB 090510 at 95% confidence [24], gives  $a < 5.3 \times 10^{-27}$  m on an axis; the same authors’ conservative limit allowing for source-intrinsic effects, 4.0  $\times 10^{10}$  GeV, gives  $a < 1.71 \times 10^{-26}$  m. For comparison, the confinement length  $\hbar c/(0.2 \text{ GeV})$  is  $9.9 \times 10^{-16}$  m. Within the same tree-level electron identification, a hadron-scale spacing is strongly excluded: at  $a = 1$  fm the threshold is 14 MeV and at  $a = 0.1$  fm it is 45 MeV. It is useful to compare these bounds with reduced wavelengths, but a wavelength comparison is not by itself an exclusion. The 4.6 TeV upper end of DAMPE’s measured combined electron–positron spectrum [25] corresponds to  $\hbar c/E = 4.3 \times 10^{-20}$  m; the inferred Crab electrons give  $8.6 \times 10^{-23}$  m. Finite range and a compact spatial Brillouin zone do not, by themselves, bound the transfer Hamiltonian:  $-\log|z|$  diverges when a transfer eigenvalue approaches zero. Neither wavelength is therefore presented as an action-independent upper bound on  $a$ .

At the Planck length, the nominal suppression factor  $(E/E_{\text{Pl}})^2$  at the Crab energy is  $3.5 \times 10^{-26}$ . This illustrates a possible scale choice; no scale is derived here. A counterterm that cancels the displayed dimension-six dispersion coefficient changes the question. Its effect on the spacing bound requires the next dispersion term, the interacting coefficients and the relevant reaction rates. No numerical improvement factor is established in this paper. On any sufficiently fine lattice a particle can be a mode spread over many cells; its internal cell labels need not set its spatial extent. For example, the electron’s reduced Compton wavelength is  $2.4 \times 10^{22}$  Planck lengths.

The estimates use the tree-level  $\kappa_f$ , identify the electron with the free PP+ pole, and assume the lattice frame moves slowly relative to ours; the radiative correction to the dimension-six coefficient was not computed. A perturbatively small correction is an additional assumption, with no bound on logarithmic or other enhancements supplied here, and the angular factor on  $c$  varies by 3/2 between axes, which is  $\sqrt{3/2}$  on  $a$ .

## 9 What the results establish

PL-1N supplies a conditional result about linearised gravity on a particular spatial lattice. Its strongest analytic result is the all-direction leading closure theorem. Its strongest numerical

result is the absence of a physical instability in the stated samples, including narrow regions that defeated the predecessor law. The imposed lapse normalisation reproduces the leading static response. The longitudinal source calculation shows why these successes should not be identified with linearised GR at finite spacing: energy continuity is not generally enforced by the geometric equations, and the time-gauge relation survives exactly only on special momentum lines in the region examined.

PP+ supplies a separate matter construction. Its ordered projector hop is explicit, its relation to Wilson fermions is algebraic, and its leading free anisotropy coefficient is derived. The common leading kinetic cone comes from the assumed four-dimensional symmetry and equal spacings. It is a useful protection mechanism within that setting, but is not a derivation of spacetime symmetry from quantum information. The construction has no exact chiral symmetry and consequently retains Wilson-type additive mass renormalisation. The word “improved” here refers to the displayed free dispersion coefficient; it does not mean that all interacting  $O(a)$  effects have been removed.

### 9.1 The missing joint theory

A shared lattice and compatible flat-space dispersion do not establish universal gravitation. That requires a single matter–gauge–geometry action whose variation supplies both the field equations and their sources, with the source identities compatible with the gravitational constraints. Matter and light must then respond to the same weak metric in a physical sector with acceptable norms. These requirements cannot be replaced by assigning each sector its own continuum stress tensor.

Recent exploratory checks of a separate four-dimensional candidate, C1, make this distinction concrete. Its formal leading smooth-field expansion has a common electron–photon metric coupling, with sources obtained by varying the action. Nevertheless its finite-spacing displacement identity fails. A subsequent fixed-coefficient check finds a gauge-invariant pole on the real Euclidean-frequency contour of its full geometry–connection covariance, obstructing that prescription at the tested finite spacings. The pole’s trajectory as the spacing tends to zero has not established a continuum obstruction. The individual graviton pole norms were not determined. C1 is not PL-1N, and these failures do not invalidate the PL-1N samples or the flat PP+ action. They do prevent the separate results being presented as a completed unified theory. The saved reports are included in the supplement; no new C1 calculation is reported here.

## 10 Limitations and research priorities

**Action and scale.** The geometry, statistics and actions are inputs. Neither Newton’s constant nor a physical lattice spacing is derived. Premise E defines the exact coefficient law inferred from a floating-point construction; MC and N choose its closure condition and source normalisation. The information-first programme still needs a physical mechanism selecting an action and a map from quantum observables and records to its geometric variables. Algebraic compatibility alone does not supply that mechanism.

**Linear gravity.** A uniform stability proof remains absent, especially on the singular kinetic surfaces and structural-null lines. The closure theorem uses the analytic-family argument stated in Appendix A; it has not been machine-formalised. No symmetry is known to protect the exact  $c = 0$  relation from changes to the action. The static and transverse source checks use declared matter rules; the matter momentum map and the continuity law of actual lattice matter remain unestablished. The breathing-source calculation is exploratory and has not been separately reviewed. It does not determine integrated radiated power or establish a lattice analogue of nonlinear Birkhoff’s theorem.

**Nonlinear gravity.** A separate auxiliary spin-2 study began with a 294-coefficient cubic family. The earlier attribution of its failures to a Wilson-like even part of the backward derivative was withdrawn: that part is a field-placement phase. A local subtraction  $V_{294} - T$  removes two off-shell diagnostics, but those diagnostics cannot distinguish the relevant field-redefinition artefact. The subsequent finite-range completion M1 fails a necessary cubic Noether identity on its actual lattice shell. Bounded  $O(a^2)$  correction families with the free kernel and gauge map fixed were also excluded by exact certificates. M1 separately violates its stipulated placement-aware linear reflection already at  $O(a)$ , which an  $O(a^2)$ -only addition cannot repair. These are results for specified auxiliary actions and finite operator families, not a general no-go theorem. No nonlinear completion of PL-1N follows from them. The supplement preserves the corrected sequence of findings.

**Fermions, positivity and interactions.** The one-step PP+ transfer matrix can have negative eigenvalues, so the Hamiltonian uses its square on non-null support. The integrated matter-gauge positivity argument remains conditional on the boundary, factorisation and gauge-measure hypotheses stated in Appendix B. It does not establish a positive measure for a dynamical geometry. The one-loop study changed its criterion after the original all-direction test failed; the remaining diagonal residual is disclosed in §8. Symmetry protects the leading invariant coefficients under the stated assumptions, but does not determine finite-mass dispersion, the higher-order coefficients, or all nonperturbative sectors. The gluon’s non-Abelian loops and the correction to the electron’s dimension-six coefficient have not been evaluated.

**Physical interpretation.** The astrophysical bound assumes the free PP+ branch describes the electron, the lattice is physical, the adopted source interpretation is applicable, and ordinary rapid vacuum-Cherenkov losses follow above threshold. It is an illustrative translation of existing observational constraints, not a new observational result or an exhaustive present-day bound. A regulator whose spacing is taken to zero has a different purpose and is not constrained in that way.

**Priority.** The central physics question is whether one justified joint action can give matter and light the same weak gravitational response with sources derived from that action. Before another numerical search, a successor must state how it avoids the known source-identity and physical-covariance failures. A bounded necessary test on actual free-shell states should come before exchange amplitudes, parameter fitting or broader phenomenology. If that gate passes, positive physical residues and a common response to massive matter and light are the next requirements. Uniform proofs of the separate-sector results remain worthwhile, but do not close this coupling gap. The existing chiral-kernel proof and apparatus branches remain frozen; this revision does not commission further calculations.

## 11 Related work

Regge introduced coordinate-free simplicial gravity, with the action  $\sum l_E \epsilon_E$  and its variation through the Schläfli identity [1]. Roček and Williams quantised it and recovered the standard continuum theory in the weak-field limit [8]. The canonical (ADM) formulation [2] and DeWitt’s supermetric [3] supply the lapse/shift structure and the kinetic term used here. Dirac’s classification of constraints [4] is the language of first and second class used in §3. The breaking of diffeomorphism symmetry in Regge calculus, and its canonical reflection as pseudo-constraints, were analysed by Bahr and Dittrich [5]. Piran and Williams gave a three-plus-one formulation of Regge calculus [6]. Gambini and Pullin’s consistent discretisations sidestep the constraint-algebra problem by letting the discrete equations of motion determine the lapse and shift [7]. The maximal-closure condition used here takes a different route: it keeps the lapse multipliers free

and requires one lapse combination to stay first class where the lattice permits, at linear order. We have not shown that this route is new beyond that comparison. The linear theory of free spin-2 fields goes back to Fierz and Pauli [9]. Deser derived Einstein self-coupling from the conserved currents of the initial massless spin-2 theory, within the local continuum assumptions of that construction [10], which is why nonlinear order is a sharp future test.

For §8: Wilson fermions and their transfer matrix are due to Wilson [11], Lüscher [12] and Osterwalder and Seiler [13]; the doubling theorem is Nielsen and Ninomiya’s [14]; the effective-theory description of lattice artefacts, under which the  $O(a^2)$  terms of (13) are irrelevant operators, is Symanzik’s [15]. Actions built for better rotational symmetry include the Brillouin operator of Dürr and Koutsou [18], with hops throughout the hypercube, and gauge theories on the body-centred hypercubic lattice, whose point group  $F_4$  is three times larger than  $B_4$  and admits no quartic invariant beyond  $(k^2)^2$  [16]; for fermions on that lattice Celmaster and Krausz found doublers that do not propagate as particles, so that the continuum theory is not Lorentz invariant [17]. The vertex-plus-link site set of the parent fermion in (12) is, in two and three dimensions, the Lieb and perovskite (edge-centred cubic) lattices of condensed-matter physics [19], there with spin-independent nearest-neighbour hops and a spin-orbit term; PP+ itself lives on the link midpoints alone. The naturalness problem of Planck-scale Lorentz violation at dimension four, against which an unbroken  $B_4$  symmetry protects the leading local kinetic terms under the assumptions of §8, was emphasised by Collins et al. [20]; the astrophysical constraints used in §8 follow the treatment of Jacobson, Liberati and Mattingly [21], with inputs from the LHAASO Crab spectrum [22] and the Fermi-LAT dispersion limits [24]; the electron bound itself was first derived from the LHAASO Crab photon by Li and Ma [23], and §8 only re-expresses it in the lattice spacing. The initial search for related link-midpoint projector actions used the web and INSPIRE titles. It was not a systematic citation search, so no claim of exhaustive novelty is made.

## A Exact certificate for leading constraint closure

**Matrices and the meaning of the certificate.** Let  $\tilde{A}_j, \tilde{G}_j, \mathcal{M}_j$  denote the coefficients of  $h^j$  in the regular series of (4) and (5). The matrices used in the certificate are

$$B_0 = [\tilde{A}_0^\dagger \mid -\mathcal{M}_0 \tilde{G}_0], \quad (15)$$

$$B_1 = [\tilde{A}_1^\dagger \mid -(\mathcal{M}_1 \tilde{G}_0 + \mathcal{M}_0 \tilde{G}_1)]. \quad (16)$$

Both are  $11 \times 8$ . For  $x_j = (w_j, \zeta_j)$ , closure to the stated order means  $B_0 x_0 = 0$ ,  $B_0 x_1 + B_1 x_0 = 0$ , with  $w_0 \neq 0$ . Thus a finite exact compatibility calculation supplies the first two coefficients of the required series; it does not assert exact closure at nonzero  $h$ .

**Relation to the numerical defect.** At fixed  $\mathbf{u} \neq 0$ , put  $C(h) = \mathcal{M}(h) \tilde{G}(h)$ . The rank lemmas give full column rank at  $h = 0$ , so the projector  $P(h) = I - C(C^\dagger C)^{-1} C^\dagger$  is real-analytic near zero. The residual after eliminating  $\zeta$  is  $F(h)w = P(h) \tilde{A}(h)^\dagger w$ . A one-parameter analytic matrix has a local Smith form under analytic changes of bases invertible at zero: its nonzero factors have the form  $h^{\nu_j}$  times analytic units. Such changes preserve vanishing orders. Consequently its smallest singular value is  $O(h^2)$  precisely when there is an analytic  $w$  with  $w(0) \neq 0$  and  $F(h)w(h) = O(h^2)$ ; an identically null column also satisfies this criterion. This explains the analytic-family step connecting the exact compatibility statement to the normalised numerical defect. The full-rank conditions and Hermiticity of  $\mathcal{M}$  are essential. The argument is stated explicitly rather than treated as another numerical test, and the certificate itself establishes the formal-series condition above.

*Proof outline.* Every certificate step is an exact computation in  $\mathbb{Q}(\sqrt{3}, i)$ .

1. *Uniform polynomial construction.* The Bloch series coefficients are homogeneous in  $\mathbf{u}$ . The closure system  $B_0(\mathbf{u})$ ,  $B_1(\mathbf{u})$  has homogeneous columns of degrees  $(2, 0, 0, 0, 0, 1, 1, 1)$  and  $+1$ .
2. *Identities from finitely many points.* Fix any 7-row set  $R$  with Cramer kernel vector  $n_0^R$ . Every  $8 \times 8$  minor of  $[B_0 \mid B_1 n_0^R]$  that uses the last column is homogeneous of degree at most 11, and closure at a direction forces these minors to vanish there. Closure is verified exactly at the 78 points  $(i, j, 1)$  with  $i + j \leq 11$ . These are unisolvent for polynomials of degree at most 11 after dehomogenisation, so the minors vanish identically. Some row set  $R$  has  $n_0^R \neq 0$  at every direction, so the identity carries content there.
3. *No exceptional directions.* Exact lemmas show that for every  $\mathbf{u} \neq 0$ :  $\mathcal{M}(0)$  has rank 11;  $\text{rank } \tilde{G}_0 = 6$  and  $\mathcal{M}_0 \tilde{G}_0$  is injective;  $\dim \ker B_0 = 1$ , and every kernel vector of  $B_0$  has a non-zero lapse part; and  $\hat{A}_0$  has rank 2. The last step uses the identity  $r_0(\mathbf{u}) \cdot D(0)\delta = -2|\mathbf{u}|^2$ , with  $D(0)$  the strain dictionary.  $\square$

[PROVED, CONDITIONAL ON PREMISE E; REVIEWED] Two controls and one review back the certificate:

- **The certificate can fail.** The non-closing control lapse  $\beta = 9/10$  fails  $O(k^2)$  closure at 55 of the 78 grid points. It closes only on the coordinate planes.
- **The certificate reproduces earlier work.** It reproduces an earlier adaptive exact computation on six integer rays.
- **Independent re-proof.** The reviewer re-proved the theorem symbolically in  $\mathbf{u}$ , with no grid, using a different field representation, different gauge and staggered-row choices, and a treatment of the exceptional directions that avoids  $D(0)$ . The closure identity holds for 35 of 35 row sets at  $R^*$  and at Route 3, and for none at  $\beta = 9/10$ . No counterexample was found at 12 exact off-grid integer directions or at 6 irrational directions.

## B Action conventions and the reflection construction

This appendix specifies the actions used in the saved calculations. The accompanying `evidence/law_exact.json` and the source files indexed by `EVIDENCE_MANIFEST.json` are unchanged copies of existing evidence. They are supplied to make the definitions inspectable; no model was rebuilt for this revision. The separate three-dimensional gravity and four-dimensional matter-gauge actions are not combined here.

### B.1 Gravity variables, coefficients and signs

The spatial builder uses doubled coordinates. In physical lattice units its C and K vertices are at  $\mathbf{R}$  and  $\mathbf{R} + (1, 1, 1)/2$ , respectively. The stored edge coordinate is  $r_E = 2l_E/a$ , and  $q_E = \delta r_E$ ; thus the reference lateral and equatorial lengths in these coordinates are  $\sqrt{3}$  and 2. For a uniform strain,  $q = D(0)h$  in the saved dictionary, with strain columns ordered as  $(h_{11}, h_{22}, h_{33}, h_{12}, h_{23}, h_{13})$ . Converting  $q$  to a physical length change requires the factor  $a/2$ .

The eleven edge components are ordered first as the equatorial classes  $(E, 0), (E, 1), (E, 2)$ , then as  $(L, s_1, s_2, s_3)$  for the eight sign triples, in lexicographic order with  $-1$  before  $+1$ . A lateral class joins a C vertex to the K vertex at displacement  $(s_1, s_2, s_3)/2$ . The six displacement columns are  $(\xi_{C1}, \xi_{C2}, \xi_{C3}, \xi_{K1}, \xi_{K2}, \xi_{K3})$ . The matrices use the unit-cell Bloch phase  $e^{i\mathbf{k} \cdot \mathbf{d}}$ , where  $\mathbf{d}$  is the integer difference of the stored cell labels; intra-cell positions are part of the dictionaries, not an additional phase in these coefficient lists.

The exact table has three lists: **Mc**, **hc** and **gc**, representing  $\mathcal{M}$  at  $C_R = 1$ ,  $H_R$  and  $G$ . An entry  $(r, c, \mathbf{d}, n, b)$  contributes

$$X_{rc}(\mathbf{k}) += \frac{n}{192}(\sqrt{3})^b e^{i\mathbf{k}\cdot\mathbf{d}}, \quad b \in \{0, 1\}. \quad (17)$$

Row and column indices are zero-based. Repeated entries are added, not overwritten. The lists contain 432, 432 and 66 entries, respectively. The original 952-entry identification also included the predecessor's 22 vertex-lapse coefficients; the three shared matrix lists therefore contain 930 entries. PL-1N replaces that lapse prescription by the orbit lists **coef.C** and **coef.K** in the same exact table and the weights  $R^*$ . Each orbit-list entry  $(c, \mathbf{d})$  contributes  $e^{i\mathbf{k}\cdot\mathbf{d}}$  to its row of  $\Phi_C$  or  $\Phi_K$ . This completely specifies  $A$  through (1). The table preserves the original numerical identification as premise E; copying it does not supply an independent geometric derivation.

With  $K = -C_R H_R$  and  $\mathcal{M} = C_R \mathcal{M}|_{C_R=1}$ , the source-free quadratic Lagrangian is

$$L^{(2)} = \int_{\text{BZ}} \frac{d^3 k}{(2\pi)^3} \left[ \frac{1}{2} (\dot{q} - Gs)^\dagger \mathcal{M} (\dot{q} - Gs) - \frac{1}{2} q^\dagger K q - \text{Re}(n^\dagger A q) \right]. \quad (18)$$

Here  $s$  and  $n$  are the six shift and two lapse perturbations, with the Fourier reality condition understood. The source code uses  $C_R = 1$ ; a positive overall  $C_R$  fixes normalisation without changing the free frequencies. In particular the stiffness is  $-C_R H_R$ , not  $+C_R H_R$ . Varying  $n$  gives  $Aq = 0$  and varying  $s$  gives the momentum constraint  $G^\dagger \mathcal{M} (\dot{q} - Gs) = 0$ . The source-free action has the exact spatial transformation  $q \mapsto q + G\xi$ ,  $s \mapsto s + \dot{\xi}$ , using  $H_R G = A G = 0$ . Additional lapse gauge freedom requires the closure condition of §3. The source couplings of §6 are additional premises, not a stress tensor derived by varying a PP+ action with respect to these variables.

The axis-4 kinetic term assembles the tetrahedral forms  $\frac{C_R}{2} \sum_t V_t [\text{tr}(h_a h_b) - \text{tr}(h_a) \text{tr}(h_b)]$ , where  $h_a$  is the linear metric change in tetrahedron  $t$  due to unit change of its edge coordinate  $q_a$ . The four tetrahedra meet at the apex–apex diagonal, whose variation is fixed by the cell construction rather than added as an independent degree of freedom. The saved snapshot specifies this map by both the rigidity and length-based constructions. For the exact law used in the certificate, the coefficient table above is the defining object.

## B.2 The compact gauge action and its quadratic kernel

Orient each bridge once. If its ends are  $a, b$ , let  $U_{ba} = U_{ab}^\dagger$ , and use the ordinary fundamental trace, with  $N_c = 1$  for  $\text{U}(1)$  and  $N_c = 3$  for  $\text{SU}(3)$ . These groups were tested separately. For each elementary plaquette  $P = (c_0, \dots, c_{n-1})$  surrounding a ridge of the cell complex, set

$$U_P = U_{c_0 c_1} \cdots U_{c_{n-1} c_0}, \quad S_G = \sum_P \beta_P (N_c - \text{Re tr } U_P). \quad (19)$$

Each unoriented plaquette is included once; reversal leaves the real trace unchanged. The saved action-invariance code evaluates  $\sum_P \beta_P \text{Re tr } U_P$ , which differs from  $-S_G$  only by a constant. No division of the trace by  $N_c$  is made in this convention.

For the chosen time axis, a T cell lies on a time-directed bond of the parent grid; the remaining cells are S cells. The plaquette classes and baseline weights are

| Class | Cycle                          | $\beta_P$ |
|-------|--------------------------------|-----------|
| ST    | triangle with no T cell        | 2         |
| TT    | triangle with one T cell       | 2         |
| SQ    | quadrilateral with no T cell   | 2         |
| TQ    | quadrilateral with two T cells | 2         |

The names depend on the chosen time axis; equal baseline weights respect  $B_4$ . A general reweighting need not. The incidence matrices in a unit cell are  $d_0 : 4 \rightarrow 24$  and  $d_1 : 24 \rightarrow 38$ , with signed entries and Bloch phases set by the oriented bridges and plaquette cycles. In the  $U(1)$  link-angle convention  $U_b = e^{iA_b}$ ,

$$S_G^{(2)} = \frac{1}{2} A^\dagger K_\gamma A, \quad K_\gamma = d_1^\dagger \text{diag}(\beta_P) d_1, \quad d_1 d_0 = 0. \quad (20)$$

The saved discrete-exterior-calculus weights are  $\star_1 = 1/3$  and  $\star_2 = 2$  on every plaquette class. The photon recount imports exactly the baseline weights and incidence builder above. It uses  $K_{\text{tot}} = K_\gamma + \star_1^2 d_0 d_0^\dagger$  and the site-ghost kernel  $\star_1^2 d_0^\dagger d_0$ , removing the gauge and ghost factors before quoting the physical light roots. The photon-mass regulator belongs to the loop calculation, not this free-root calculation. The loop photon propagator instead inverts

$$K_\gamma + \frac{\star_1^2}{\alpha} d_0 d_0^\dagger + \mu^2 \star_1 I_{24}, \quad \mu^2 = 0.01. \quad (21)$$

The quoted loop run has  $e^2 = 1$ , with  $U_b = e^{ieA_b}$  in the vertices,  $\alpha = 1$  and an  $\alpha = 2$  check. No claim about a regulator-removal limit is attached to that run.

The fermion action in these fixed gauge variables is

$$S_F = \sum_c M \bar{\psi}_c \psi_c + \sum_{\langle ab \rangle} \left[ \bar{\psi}_a (V_{ab} \otimes U_{ab}) \psi_b + \bar{\psi}_b (V_{ba} \otimes U_{ab}^\dagger) \psi_a \right], \quad M = m + 4, \quad (22)$$

where  $V$  is PP+ in (10),  $\kappa = 2/3$ , and the sum includes each bridge once. The gauge measure is the product Haar measure, with Boltzmann weight  $e^{-S_G - S_F}$  and Grassmann integration for the fermions. This is an action on the prescribed four-dimensional geometry.

### B.3 Reflection geometry, factorisation and its qualifications

The time-cell reflection is  $\theta_T : t \mapsto 1 - t$ , with fixed T-cell planes at  $t = 1/2$  and  $t = (1 + N_t)/2$  on an even- $N_t$  torus. The saved finite test used  $(L_s, N_t) = (3, 6)$ , periodic spatial boundaries,  $m = 0.1$ , and antiperiodic fermion time boundaries. The minus-sign twist is placed on the bridges immediately above the far fixed plane. This placement was calibrated against the Wilson-chain and free-PP+ controls after the original periodic-time prescription failed. A periodic fermion trace inserts  $(-1)^F$  and is not the positive thermal trace used in this test.

The boundary cells have no bridges to each other; every bridge belongs to one half and its reflection belongs to the other. Every plaquette meeting both open halves is a quadrilateral of the form  $(c, T, \theta_T c, T')$ , with  $T, T'$  on a fixed plane. These incidence statements were checked in the saved torus builder. They identify the factors required by the reflection construction; the finite checks alone are not a proof for every possible boundary geometry.

Let  $\mathcal{A}_+$  be the algebra generated by fields and link functions on the positive half, with the boundary fields treated by their specified integration. The antilinear reflection exchanges  $\psi$  and  $\bar{\psi}$  with  $\gamma_4$ , reverses Grassmann-product order, and reflects the oriented gauge paths. Positivity is the non-negativity of the reflected form  $\langle \Theta F F \rangle$  on the appropriate graded observable algebra, or on its gauge-invariant even subalgebra; null vectors are then quotiented out. The particle and antiparticle Gram conventions in the saved test are fixed by the known-positive controls. For a fixed-background test the compatible prescription is  $U_{\theta_T a \rightarrow \theta_T b} = U_{a \rightarrow b}$  on oriented pairs, with reversed links given by the adjoint. Complex conjugation is already part of  $\Theta$  acting on functions; applying it to the fixed background a second time gives the failed comparison prescription. Gauge integration itself does not restrict the gauge fields to reflection-invariant configurations.

On each boundary T cell the fermion block is  $MI$ . Integrating that block out gives its determinant factor  $M^{4N_c}$  and the Schur term

$$D_{S_0 S_1}^{\text{eff}} = -\frac{1}{M} (V_{S_0 T} \otimes U_{S_0 T}) (V_{T S_1} \otimes U_{T S_1}). \quad (23)$$

Here  $S_0$  and  $S_1$  lie in opposite halves. The saved identities are  $(1 + \gamma_4)V_{TS_1} = 0$  and  $V_{S_0T} = \gamma_4 V_{TS_1}^\dagger \gamma_4$  for reflected bonds. With the registered Grassmann reflection convention, they put the cross-plane contribution in the Osterwalder–Seiler factorised form, schematically  $\sum_T \Theta(\chi_T) \chi_T / M$ , where  $\chi_T = \sum_{S_1} (V_{TS_1} \otimes U_{TS_1}) \psi_{S_1}$ . The boundary determinant is retained. The algebraic step requires  $M > 0$ ; it is not a numerical scan over all masses or an interacting phase diagram. The tested positive bare mass satisfies this condition.

For the gauge factors, the invoked hypothesis is a non-negative character expansion

$$e^{\beta \text{Re tr } U} = \sum_{\rho} c_{\rho}(\beta) \chi_{\rho}(U), \quad c_{\rho}(\beta) \geq 0, \quad (24)$$

for  $U(1)$  or  $SU(3)$  with the ordinary Wilson weight,  $\beta \geq 0$ , and Haar integration. A crossing quadrilateral splits into a path in the positive half and its reflected path. Unitary representation matrix elements then express its character as reflected products with the coefficients in (24). Together with the fermion factorisation, reflection-paired same-half terms and the stated boundary integration, this is the application of the Osterwalder–Seiler construction being invoked [13]. The general character-expansion result is cited, not re-proved here. The saved evidence supplies the local projector identities, the finite incidence checks and the background Grams; it does not contain a separate complete proof of the integrated construction for all volumes and parameters. The integrated statement is therefore retained as a conditional proposition with these hypotheses, not promoted to a new theorem.

Time-cell positivity supports a self-adjoint transfer construction under those hypotheses; it does not make the one-step transfer positive. On non-null support the energy prescription is  $H = -(2a_t)^{-1} \log(T^2)$ , consistently with the sampled negative one-step eigenvalues. None of this establishes positivity after coupling to dynamical geometry or descent of the framework’s chiral ledger through the integrated physical-state quotient. That separate proof branch remains open and frozen. Finally, the finite torus and time-boundary prescription used here differ from the symmetric momentum grid used in the one-loop comparison; no claim of full  $B_4$  symmetry of that finite-temperature boundary prescription is made.

## C Numerical validation and the registered verdicts

**The registered spin-content test.** The pre-registered test of this content (G7) returned FAIL. Its numerical-rank rule (threshold  $10^{-9}$ ) misread an  $O(k^2)$  near-dependency of the constraint system as a rank deficiency on every off-axis ray. On 2 of the 10 rays this produced a speed failure,  $c^2 - 1 \approx -1.1 \times 10^{-6}$ . With the structurally determined rank, all rays pass. That correction was post hoc, so an independent reviewer pre-registered a test on fresh rays: 12 random rays and 4 small tilts from [110], with the rank fixed at 10 and decided at 50 digits.

- All 16 rays pass, with  $c^2 - 1$  between  $-5.45 \times 10^{-10}$  and  $5.14 \times 10^{-8}$  in the window fit.
- The massive mode is at  $\omega = 3.9999219$ .
- The  $\beta = 9/10$  control fails on 11 of the 12 random rays. It fails through the massive-mode criterion, so it shows the test’s power for the massive mode, not for  $c^2$ .

The registered verdict stays FAIL and is reported here, alongside its out-of-sample corroboration. [REVIEWED]

**Direct sample at  $R^*$ .** A direct sample at  $R^*$  found no physical instability (no growth and no negative reduced energy) at any point. [SAMPLED; REVIEWED]

- **Double precision.** 9,704 points: random zone points, tilt scans and fans about  $\langle 110 \rangle$ , spheres about the high-symmetry points  $(\pi, \pi, 0)$ ,  $(\pi, \pi, \pi)$  and  $(\pi, 0, 0)$ , and crossings of  $\det \mathcal{M} = 0$ . The test used two reduction routes, which are one reduction in two gauges and so not independent methods, and a bordered-pencil inertia test, which is an independent method.
  - No point is unstable by both routes, and none by the bordered-pencil test.
  - The configuration route alone flagged 162 points where the routes disagree. All of them are on the corner spheres, at  $\det \mathcal{M} = 0$  or at the zone boundary, where that route’s inverse of  $\mathcal{M}$  fails.
  - It also flagged 12 points where the other route is unavailable. The bordered-pencil test passes all of these.
- **80 digits.** 295 points. These include 216 tilts from  $\langle 110 \rangle$  ranging from  $10^{-9}$  down to  $10^{-22}$  rad, which show no ghost. On the exact  $\langle 110 \rangle$  lines, all 18 sampled points are stable after the structural quotient.
- **The test has power.** The same 80-digit test, applied not to  $R^*$  but to the parent representative stored in double precision ( $c \approx 5 \times 10^{-16}$ ), finds ghosts at 105 of those 216 tilt points, including all 72 at tilts of  $10^{-20}$  rad and below.

Independently, the parent representative was scanned at 25,476 points by two routes. That scan’s registered verdict was UNDECIDED: 22 points carried flags from one route only, which a post-hoc 50-digit check resolved. A review found that 457 of its stable points rest on one route alone, 60 of them exactly on  $\langle 110 \rangle$ , where the frozen registration did not permit a pass. The reviewer then added 478,651 bordered-pencil evaluations, with no symmetry assumed in part of the sample, and 60–100-digit probes at the zone corner, the  $\det \mathcal{M} = 0$  surfaces and small  $|\mathbf{k}|$ . Neither found any instability.

## D Reproducibility and provenance

**Evidence availability.** The reproducibility archive is publicly available at <https://doi.org/10.5281/zenodo.23458463> (evidence release 1). It contains the cited evidence and identified execution inputs: manuscript sources, study and review packages, saved outputs, scripts and witnesses, with provenance hashes and reproduction instructions. Its `README.txt`, `EVIDENCE_INDEX.txt` and `PATH_MAP.json` locate the material, including the repository-relative paths below. The archive preserves revision 5; the source and diff for this evidence-availability correction accompany the present paper’s Zenodo version. A fresh extraction passed file-hash and numerical-quotation checks; full numerical replay and the supplied container recipe were not independently validated for the release.

**Code.** The project repository is <https://github.com/dgedge/octahedrons>. The gravity packages are in commit `cc7790772c7f284d9c11f6656ad76fae023507c3`; the fermion packages (O2a’, O2b, R1, R1r), the retrospective control scripts, the paper and its review banks were uncommitted in the revision-3 provenance record. Their deposited versions are identified by the archive manifest and hashes; public deposition does not change that historical Git provenance. Each package’s NOTE gives its command, and its transcript records the start time, the input hashes and the runtime.

- Closure theorem: `technical_notes/wp3_closure_all_directions_2026-10-15/closure_proof.py`.

- Exact-closure lemma and singular locus: `technical_notes/gravity_c3_route3_why_c_zero_2026-10-15/why_c_zero.py`.
- Spectrum: `technical_notes/wp3_provisional_law_PL1N_adoption_checks_2026-10-15/adoption_checks.py` and `technical_notes/wp3_provisional_law_PL1N_2026-10-15/pl1n_tests.py`.
- Zone stability: `technical_notes/gravity_c3_route3_nearcand0_whole_zone_2026-10-15/whole_zone_scan.py`.
- Source couplings: `technical_notes/wp3_newtonian_source_2026-10-15/newtonian_source.py` and `technical_notes/wp3_test_particle_2026-10-15/test_particle.py`.
- Longitudinal sector: `technical_notes/wp3_longitudinal_shift_2026-10-16/longitudinal_shift.py`.
- PL-1: `technical_notes/wp3_provisional_law_PL1_2026-10-14/`, including the hash-pinned Route 3 snapshot.
- Fermion class and PP+ (§8): `technical_notes/spacetime_02a_prime_bond_class_2026-10-17/o2a_prime.py`; gauge coupling and one loop: `technical_notes/spacetime_02b_gauge_coupled_2026-10-17/o2b.py` (the positivity numbers come from the second run's script version, preserved as `run2_as_registered/o2b.py`; the current script reproduces them exactly); dispersion coefficients: `technical_notes/st4d_R1_free_spectra_2026-10-17/r1_spectra.py` and `technical_notes/st4d_R1r_photon_recount_2026-10-17/r1r_recount.py`; the Wilson map, the determinant identity, the closed-form extraction, the  $B_4/F_4$  invariant counts and the scale estimates: `technical_notes/programme_retrospective_2026-10-18/pp_plus_vs_wilson_control.py`, `f4_invariants_control.py` and `liv_bound_estimate.py` in the same directory, with their output transcripts.
- Review of §8: `technical_notes/paper_P1_section8_fermion_REVIEW_2026-10-18/`, two fresh-context reviewers (replay and verifier audit; mathematics and literature) with their reports, scripts and outputs; the photon numbers attributed to the review are read from `reviewer_B/out/b5_photon.out`.

The fermion packages have no review packages of their own; the review of §8 replayed them bitwise. Their standing is exploratory throughout.

**Reviews.** The review packages sit beside the studies:

- `wp3_closure_all_directions_REVIEW_c5da7b91_2026-10-15;`
- `wp3_provisional_law_PL1N_REVIEW_c5da7b91_2026-10-15;`
- `gravity_c3_route3_nearcand0_whole_zone_REVIEW_2026-10-15;`
- `wp3_PL1N_Q1_REVIEW_2026-10-15;`
- `wp3_newtonian_testparticle_REVIEW_2026-10-15;`
- `wp3_provisional_law_PL1_REVIEW_ceb88a35_2026-10-14.`

The quoted numbers in this paper are checked against those records by `verify_numbers.py` in this paper's directory.

**Method and its limits.** The studies were constructed and reviewed with AI coding agents under a pre-registration protocol:

- an outcome map was written before each script;
- each script asserts the numbers it reports and includes negative controls;
- post-hoc steps are labelled as post hoc.

The reviewing agents worked in separate contexts but were briefed by the constructing sessions. Their reviews provide reproduction and criticism, with limited independence; they are not external peer review. AI assistance was also used in revising the exposition and checking citations. The author directs the work and is responsible for verifying the final text, calculations and references.

**Revision history.** Revision 5 applied the final review of the readable revision 4. Its archived paper package preserves the preceding source, PDF and review in `inputs/`, with `REVIEW_RESPONSE.txt`, a source diff and an evidence manifest recording the corrections and the existing sources used for Appendix B. That package supplies the exact gravity table as `evidence/law_exact.json`; the broader evidence and dependencies are in the Zenodo archive cited above. Revision 6 updates evidence availability and publication status only. The scientific text, equations, numerical quotations and literature references are unchanged from revision 5. The quantitative verifier checks transcriptions against saved records; passing it does not validate a number’s physical interpretation. Its retained `liv_floor_*` identifiers label wavelength comparisons, not independent spacing bounds. No physics script or frozen research branch was run for this revision.

**Later negative results.** The `supplement/` directory contains the saved M1 on-shell Noether, bounded correction and enlarged-family certificate reports; and the C1 source, continuum and long-distance covariance reports. These are explicitly exploratory, constructor-checked findings. They are included to make the discussion current and to delimit the separate-sector claims, not to imply that their actions are PL-1N. The Zenodo archive includes the underlying scripts, witnesses and identified dependencies as well as these summaries. No frozen research branch was reopened for this revision.

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