Emergent Minimal Gauge Coupling from C_{4v} Symmetry Reduction at the TCH Vertex Figure

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Abstract

We analyse the symmetry structure of the continuous-time quantum walk on the 4.8.8 (truncated square) Archimedean tiling — which in the framework's canonical $\mathbb{Z}^3 \otimes Q_3$ (Truncated Cubic Honeycomb) substrate appears as the *vertex figure* at each TCH lattice vertex (the local C_{4v} point-group reduction), and equivalently as the coordinate-plane slice of the 3D substrate — whose bifurcated band structure was recently characterised via OTOC echo protocols [1].[1] Decomposing the 8×8 Bloch Hamiltonian into irreducible representations of the C_{4v} point group at the Γ point, we show that the gapped slow branch transforms as the fully symmetric scalar representation A_1, while the gapless fast branches transform as the two-dimensional vector representation E. At Γ, the Wigner-Eckart theorem enforces strict orthogonality between these sectors: $\langle A_1|\mathcal{H}|E\rangle = 0$. Away from the Brillouin zone centre, the momentum-induced symmetry reduction $C_{4v} \to C_s$ breaks this exact selection rule. We compute the inter-branch matrix element exactly and find $\mathcal{M} = -\frac{i}{2}\sin k_x$ for the E_x component, yielding a momentum-linear vertex $\mathcal{M} \propto i(\mathbf{k}\cdot\mathbf{E})$ that is structurally identical to the $\mathbf{p}\cdot\mathbf{A}$ minimal coupling of quantum electrodynamics. By promoting the tight-binding hopping amplitudes to dynamical $U(1)$ link variables in a Wilson lattice gauge theory, we demonstrate via a weak-field expansion that the gauge-covariant Hamiltonian natively reproduces this identical vertex. Furthermore, we identify a geometrically locked two-coupling structure (β_4, β_8) strictly governed by the tiling's silver-ratio geometry. Finally, we formulate the *velocity-unification conjecture*—proposing that renormalisation group flow drives the distinct bare lattice group velocities toward a common infrared fixed point (macroscopic Lorentz invariance)—as a concrete numerical programme for dynamical lattice Monte Carlo simulation.

1 Introduction

In standard Quantum Field Theory (QFT), the minimal gauge coupling vertex $\mathcal{H}_{\text{int}} = -e(\mathbf{p}\cdot\mathbf{A})$ is traditionally treated as an axiomatic necessity. It is bolted onto the free Dirac or Klein-Gordon equations from the top down by demanding continuous local $U(1)$ phase invariance. However, in inherently discrete informational frameworks and topological tensor networks, continuous interactions cannot be seamlessly postulated; they must emerge dynamically and structurally from the underlying discrete topology of the lattice.

In this paper, we demonstrate that the fundamental minimal coupling vertex of quantum electrodynamics (QED) requires no continuous postulates. Instead, it is strictly forced by the

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[1] In the canonical 3D framework the 4.8.8 tiling is not the primary geometric stage but a 2D-reduction layer at the TCH vertex; the C_{4v} derivation that follows is therefore the vertex-figure-level analysis of the full 3D theory. The minimal-coupling result of this paper is anchored in the framework's canonical reference document as the algebraic reduction to the QED Lagrangian.

crystalline point-group symmetry breaking of the 4.8.8 vertex figure at each TCH lattice vertex. When a quantum wavepacket propagates through the discrete network, the resulting macroscopic momentum ($\mathbf{k} \neq 0$) spontaneously breaks the full C_{4v} point-group symmetry of the unit cell. This geometric reduction fundamentally hybridises the massive scalar modes (matter on the Q_3 matter cells) with the gapless vector modes (gauge bosons on the L(TCH) line graph). We formally prove that the inter-band coupling dynamically emerging from this rigorous symmetry reduction assumes the exact structural form of the QED minimal coupling vertex.

2 The Free Theory: Recap of the Birefringent Substrate

In Ref. [1], we established that the continuous-time quantum walk $U(t) = \exp(-it\mathcal{H})$ on the 4.8.8 tiling exhibits a bifurcated operator-spreading light cone. The lattice intrinsically supports two distinct bare group velocities, $v_{\text{fast}} = 0.81$ and $v_{\text{slow}} = 0.60$ shortest-path hops per unit time, a phenomenon analogous to optical birefringence.

The algebraic foundation for this kinetic bifurcation is the topology of the Bloch Hamiltonian. In the present work, we adopt the 8-node matter octagon (C_8) as the conventional unit cell, in contrast to the bipartite two-square primitive cell utilised in Ref. [1]. While both bases yield physically identical energy spectra, they differ geometrically in their assignment of Peierls momentum phases to spatial edges. This mapping results in a basis-dependent rescaling of the bare matrix element prefactor (e.g., $-i/2$ versus $i/\sqrt{2}$). Crucially, this geometric scaling is seamlessly absorbed into the continuum normalisation of the gauge coupling via the effective lattice parameter a, leaving all macroscopic physical observables strictly invariant.

The momentum-space Hamiltonian naturally partitions into a static internal cyclic block $\mathcal{H}_{\text{int}}$ governing the octagon, and a momentum-dependent external block $\mathcal{H}_{\text{ext}}(\mathbf{k})$ governing the square gauge bridges. Diagonalisation yields eight bands separating cleanly into fast inner branches and slow outer branches. Throughout Sections 3 and 4 of this paper, we operate exclusively within this non-interacting free-theory Hamiltonian to derive the interaction vertex purely from geometric symmetry. All velocities discussed are bare ultraviolet lattice quantities; continuous gauge fields do not appear until Section 5.

3 C_{4v} Irrep Decomposition at Γ

At rest (the Brillouin zone centre, $\mathbf{k} = 0$), the full Bloch Hamiltonian $\mathcal{H}(0)$ possesses exact C_{4v} point-group symmetry. We systematically decompose the 8-dimensional Hilbert space of the unit cell into the irreducible representations (irreps) of C_{4v}:

1. **The Scalar Branch (A_1):** The fully symmetric eigenstate transforms as the one-dimensional trivial representation. It physically represents the massive, slow-moving "matter" sector.

2. **The Vector Branch (E):** A doubly degenerate, two-dimensional representation transforming identically to the orthogonal spatial dipoles (x, y). It structurally represents the gapless "gauge" sector.

3.1 The Topological Selection Rule

At exactly $\mathbf{k} = 0$, the static Hamiltonian commutes with all symmetry operators of C_{4v}. Because the A_1 and E states belong to categorically distinct irreducible representations, the Wigner-Eckart theorem enforces strict orthogonality upon the matrix elements:

$$\langle A_1 | \mathcal{H}(0) | E \rangle \equiv 0. \tag{1}$$

At $\mathbf{k} = 0$, the scalar matter sector and the vector gauge sector are perfectly decoupled. No matrix element of the Hamiltonian connects them. This is not a low-energy phenomenological approximation, but an exact geometric selection rule mathematically forbidding interaction.

4 Compatibility Reduction and the Emergent Vertex

4.1 Symmetry Reduction at Generic k

When a quantum wavepacket acquires finite crystal momentum—for instance, propagating along the principal axis $\mathbf{k} = (k_x, 0)$—it inherently breaks the full isotropic C_{4v} symmetry. The relevant spatial symmetry subduces to the "little group" of $\mathbf{k}$, which is exactly C_s (containing only the identity and a single mirror plane σ_v perpendicular to the momentum vector).

Under the standard mathematical compatibility relations governing the $C_{4v} \to C_s$ group subduction, the two-dimensional vector irrep E decomposes into $A' \oplus A''$. Crucially, the fully symmetric scalar A_1 also directly subduces to A'. Because both the massive scalar branch and the longitudinal vector component E_x now transform identically under the surviving A' representation within the little group, state mixing becomes rigorously allowed. The topological protection previously enforced by the Wigner-Eckart theorem vanishes, and the modes are dynamically forced to hybridise.

4.2 The Momentum-Linear Vertex

We explicitly compute the exact transition matrix element $\mathcal{M}_x = \langle A_1|\mathcal{H}_{\text{ext}}(k_x, 0)|E_x\rangle$. As derived in Appendix A using the full 8×8 Bloch Hamiltonian, this topological cross-wiring evaluates exactly to:

$$\mathcal{M}_x = -\frac{i}{2}\sin k_x. \tag{2}$$

In the long-wavelength continuum limit ($k_x \ll 1$), the sine function linearises to $\sin k_x \approx k_x$. Generalising to an arbitrary 2D momentum vector $\mathbf{k}$, the inter-branch coupling assumes the exact asymptotic form:

$$\mathcal{M} = -\frac{i}{2}(k_x\hat{E}_x + k_y\hat{E}_y) \propto i(\mathbf{k}\cdot\mathbf{E}). \tag{3}$$

Because physical continuum momentum is defined as $\mathbf{p} = \hbar\mathbf{k}$, and the E-branch structurally acts as the emergent vector potential field $\mathbf{A}$, this geometrically derived coupling maps identically to the standard QED minimal coupling vertex:

$$\mathcal{H}_{\text{int}} \propto i(\mathbf{p}\cdot\mathbf{A}). \tag{4}$$

We emphasise that the functional form $\mathbf{k}\cdot\mathbf{E}$ is dictated generically by the Wigner-Eckart theorem and $\mathbf{k}\cdot\mathbf{p}$ perturbation theory [2]. Any arbitrary lattice sharing the C_{4v} point group and possessing adjacent A_1 and E bands will natively produce an inter-band coupling of this form. What makes the 4.8.8 tiling phenomenologically unique is its native physical partition: the massive (gapped) branch naturally carries A_1 and the gapless branches naturally carry E—a highly specific sector distribution that does not natively occur on the standard $z = 4$ generic square lattice.

5 Wilson Gauge Theory at the 4.8.8 Vertex Figure

The preceding kinematic $\mathbf{k}\cdot\mathbf{p}$ vertex establishes a structural correspondence derived entirely from global C_{4v} symmetry reduction. To formally elevate this geometric framework to a dynamically interacting gauge theory, we promote the fixed tight-binding hopping amplitudes to dynamical $U(1)$ phase variables $\theta_{ij} \in [0, 2\pi)$ via the standard Wilson lattice formulation [3, 4].

5.1 Two-Plaquette Gauge Action and the Geometric Lock

The 4.8.8 Archimedean tiling possesses two geometrically distinct plaquette species: squares (C_4) and octagons (C_8) [8]. A naive construction of the Wilson action would necessitate two independent bare gauge couplings, β_4 and β_8. If these continuous parameters were freely adjustable, the theory would lose its fundamental parameter-free structural rigidity.

However, to faithfully reproduce the isotropic continuum Maxwell action $\int \frac{1}{2}B^2\,\mathrm{d}^2x$, the macroscopic magnetic energy density must be uniform across the spatial substrate. Because the discrete magnetic flux Φ_p strictly scales with the physical area of the given plaquette, matching the continuum area integral mathematically requires that the bare Wilson couplings scale inversely with the geometric area: $\beta_p \propto 1/\mathcal{A}_p$.

Assuming a fundamental unit edge length $a = 1$, the area of the square is exactly $\mathcal{A}_4 = 1$, and the geometric area of the regular octagon is $\mathcal{A}_8 = 2(1+\sqrt{2})$. Thus, the bare gauge couplings are permanently structurally locked:

$$\frac{\beta_8}{\beta_4} = \frac{\mathcal{A}_4}{\mathcal{A}_8} = \frac{1}{2(1+\sqrt{2})} = \frac{\sqrt{2}-1}{2} \approx 0.2071. \tag{5}$$

Gauge fluctuations within the octagonal matter sectors are strictly topologically suppressed relative to the square gauge bridges. This exact suppression ratio is dictated mathematically by the geometry of the silver ratio ($\delta_S = 1+\sqrt{2}$). Consequently, zero new continuous phenomenological parameters are introduced into the theory.

5.2 Weak-Field Expansion and Vertex Verification

Writing the continuous link variables as $\theta_{ij} \approx aA_\mu(x_i)$ along the local spatial edge direction μ, the gauge-covariant Hamiltonian expands to first order in the vector potential $\mathbf{A}$ as:

$$\mathcal{H}_{\text{gauge}} \approx \mathcal{H}_{\text{free}} - ia \sum_{\langle i,j\rangle} A_\mu \left(c_i^\dagger c_j - c_j^\dagger c_i\right) + \mathcal{O}(A^2). \tag{6}$$

As explicitly detailed in Appendix B, Fourier-transforming the $\mathcal{O}(A)$ term into the momentum-space Bloch basis and projecting it onto the A_1 and E irreps at small $\mathbf{k}$ yields an effective coupling of the exact form:

$$\mathcal{H}_{\text{int}} \propto \psi_{A_1}^\dagger (\mathbf{A}\cdot\mathbf{E})\psi_E + \text{h.c.} \tag{7}$$

This is structurally identical to the inter-band vertex derived independently from the free-theory $\mathbf{k}\cdot\mathbf{p}$ expansion. This convergence constitutes a rigorous, Ward-identity-level consistency check: the inter-band coupling that naturally emerges from spatial point-group symmetry reduction is identically the exact weak-field limit of the rigorous lattice gauge coupling.

6 Discussion: The $E = +1$ Spectral Resonance

In Ref. [1], we established that coherent macroscopic wavepacket transmission T across the C_4 gauge bridge diverges precisely at the energy eigenvalue $E = +1$, representing an absolute geometric pole of perfect transmission. By algebraically diagonalising the explicit 8×8 Bloch Hamiltonian (Appendix A), we find that the gapless vector doublet (E_x, E_y) resides precisely at this exact eigenvalue $\lambda = +1$.

The $E = +1$ degeneracy point—where the macroscopic transmission pole and the point-group irrep dissolution strictly mathematically coincide—acts as a highly natural topological candidate for the fundamental gauge-mediated interaction scale. We highlight this striking structural alignment here, but defer its formal field-theoretic calculation to subsequent analysis of the fully interacting gauge lattice.

7 Limitations and Scope

To guarantee rigorous theoretical clarity, we explicitly bound the scope of the current analytical work:

1. The free-theory vertex derived in Section 4 is a $\mathbf{k} \cdot \mathbf{p}$ result describing inter-band mixing under a global C_{4v} symmetry reduction. It establishes a profound structural correspondence with QED minimal coupling, rather than a direct derivation of local $U(1)$ gauge invariance (which is explicitly constructed subsequently in Section 5).

2. The $U(1)$ Wilson gauge theory formulated in Section 5 is analytically defined but has not yet been dynamically solved. Determining its exact phase diagram (confining vs. Coulomb) and the static potential $V(r)$ requires explicit numerical Monte Carlo simulation.

3. The velocity-unification conjecture stated below in Section 8 currently remains unproven. We do not claim emergent Lorentz invariance as a solved analytical fact; rather, we formulate it as the central, mathematically falsifiable numerical prediction of this lattice framework.

8 Numerical Programme & Velocity Unification

On a discrete static lattice, the emergent gapless fast branch (v_{fast}, corresponding to gauge light) and the gapped slow branch (v_{slow}, corresponding to massive matter) possess fundamentally distinct bare group velocities [5]. Consequently, macroscopic Lorentz invariance is maximally broken at the tight-binding ultraviolet (UV) lattice scale.

Conjecture 1 (Velocity Unification)**.** *Macroscopic Special Relativity ($v_{fast} = v_{slow} \equiv c$) is not a fundamental axiomatic input of the universe, but an Infrared (IR) Fixed Point of the Renormalisation Group (RG) flow, dynamically driven by the mutual interaction drag generated by the emergent $\mathbf{p} \cdot \mathbf{A}$ vertex.*

We frame this conjecture as a concrete, solvable challenge for the lattice Monte Carlo simulation community. The objective is to rigorously test whether the UV velocity difference $\Delta v = v_{\text{fast}} - v_{\text{slow}}$ dynamically flows to zero under RG transformations as the physical momentum scale $\mu \to 0$. If verified, this identifies the velocity splitting as a strictly irrelevant geometric perturbation that natively vanishes in the infrared.

A comprehensive numerical programme to resolve this proceeds in three stages:

1. **Pure Gauge Theory:** Extract the static potential $V(r)$ and determine if the geometrically locked two-plaquette action (Eq. 5) naturally places the compact $U(1)$ theory in the deconfining Coulomb phase.

2. **Quenched Fermions:** Compute exact fermion propagators in fixed gauge backgrounds to empirically measure the dressed velocities $v_{\text{fast}}(\beta)$ and $v_{\text{slow}}(\beta)$ as functional invariants of the coupling strength.

3. **Dynamical Fermions:** Utilise full Hybrid Monte Carlo (HMC) routines to simulate the fully interacting theory, unequivocally confirming whether mutual vacuum-polarisation drag forces the absolute condition $\Delta v \to 0$ in the macroscopic continuum limit.

9 Conclusion

The $U(1)$ minimal coupling vertex does not need to be manually postulated as an independent axiom of physics. The native C_{4v} point-group symmetry of the 4.8.8 vertex figure at each

TCH lattice vertex spontaneously isolates massive scalar branches from vector gauge branches. Spatial momentum translation fundamentally breaks this symmetry, mathematically forcing an exact $\mathbf{p} \cdot \mathbf{A}$ kinematic hybridisation. By promoting this static topological architecture to a dynamic Wilson gauge theory bound by the silver-ratio geometric coupling lock, we find that the continuous symmetries of macroscopic spacetime—from the necessity of minimal coupling to the candidate mechanisms for emergent Lorentz invariance—are intimately, naturally dictated by the discrete algorithmic geometry of the underlying TCH substrate. The bridge from the vertex-figure-level C_{4v} analysis presented here to the macroscopic SC-gauge-web isotropy of the full 3D framework runs through the velocity-unification RG flow conjectured in Section 8.

A Explicit Off-Diagonal Coupling Matrix

We explicitly derive the exact inter-branch mixing matrix element $\mathcal{M}_x = \langle A_1|\mathcal{H}(\mathbf{k})|E_x\rangle$ by constructing the 8×8 Bloch Hamiltonian for the 4.8.8 lattice. The conventional unit cell is defined as the 8-node matter octagon, indexing the internal boundary nodes $1 \ldots 8$ in a counter-clockwise orientation.

A.1 The Bloch Hamiltonian

The full Hamiltonian $\mathcal{H}(\mathbf{k}) = \mathcal{H}_{\text{int}} + \mathcal{H}_{\text{ext}}(\mathbf{k})$ consists of the internal cyclic connections spanning the octagon and the external square-bridge connections linking neighbouring cells. Normalising the internal hopping amplitude to $t = 1$, we have:

$$\mathcal{H}_{\text{int}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{8}$$

Defining the spatial orientation such that internal nodes $(1, 2)$ face $+y$, $(3, 4)$ face $-x$, $(5, 6)$ face $-y$, and $(7, 8)$ face $+x$, the momentum-dependent hopping matrix $\mathcal{H}_{\text{ext}}(\mathbf{k})$ connecting adjacent cells is given by:

$$\mathcal{H}_{\text{ext}}(\mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e^{ik_y} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{ik_y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-ik_x} \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-ik_x} & 0 \\ 0 & e^{-ik_y} & 0 & 0 & 0 & 0 & 0 & 0 \\ e^{-ik_y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{ik_x} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{ik_x} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

It is easily verified by inspection that $\mathcal{H}_{\text{ext}}(\mathbf{k})$ remains strictly Hermitian for any real momentum vector $\mathbf{k}$.

A.2 Eigenstates at Γ

At the exact Brillouin zone centre ($\mathbf{k} = 0$), all inter-cell phase factors perfectly evaluate to 1. The fully symmetric scalar matter branch (A_1) is the uniform eigenvector corresponding to the maximum intra-cell eigenvalue $\lambda = 3$:

$$|A_1\rangle = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T . \tag{10}$$

The gapless vector branch (E_x) corresponds strictly to the spatial dipole oriented along the x-axis ($\lambda = 1$):

$$|E_x\rangle = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}^T . \tag{11}$$

As mandated by the Wigner-Eckart theorem at Γ, these states are mathematically orthogonal: $\langle A_1 | E_x \rangle = 0$.

A.3 Symmetry Reduction and Coupling

We inject pure longitudinal momentum $\mathbf{k} = (k_x, 0)$. Because $\mathcal{H}_{\text{int}}$ is entirely independent of $\mathbf{k}$, dynamic coupling emerges exclusively from the external gauge bridge boundaries. Operating $\mathcal{H}_{\text{ext}}(k_x, 0)$ on $|E_x\rangle$ yields an intermediate vector $|v\rangle$. Note that $\mathcal{H}_{\text{ext}}$ couples target site j to a geometrically distant source site j' via the bridge; the values in parentheses below evaluate the components of $|E_x\rangle$ at the *source* sites j':

$$\begin{aligned} v_1 &= e^{i\cdot 0}(1) = 1, & v_2 &= e^{i\cdot 0}(-1) = -1 \\ v_3 &= e^{-ik_x}(1) = e^{-ik_x}, & v_4 &= e^{-ik_x}(1) = e^{-ik_x} \\ v_5 &= e^{-i\cdot 0}(-1) = -1, & v_6 &= e^{-i\cdot 0}(1) = 1 \\ v_7 &= e^{ik_x}(-1) = -e^{ik_x}, & v_8 &= e^{ik_x}(-1) = -e^{ik_x} \end{aligned}$$

Taking the inner product of $|v\rangle$ with the perfectly uniform dual vector $\langle A_1|$ simply requires summing these eight resultant components:

$$\sum_{j=1}^{8} v_j = (1-1) + (-1+1) + 2e^{-ik_x} - 2e^{ik_x} = 2(e^{-ik_x} - e^{ik_x}) = -4i \sin k_x. \tag{12}$$

Applying the 1/8 combined normalisation factor (arising from the product of the two independent $1/\sqrt{8}$ state vectors), we recover the exact analytical coupling element:

$$\mathcal{M}_x = \langle A_1 | \mathcal{H}_{\text{ext}}(k_x, 0) | E_x \rangle = -\frac{i}{2} \sin k_x. \tag{13}$$

B Weak-Field Expansion and Ward Identity Check

To formally verify the structural correspondence between the derived geometric momentum vertex ($\mathbf{k}\cdot\mathbf{E}$) and the dynamic gauge coupling ($\mathbf{p}\cdot\mathbf{A}$), we expand the gauge-covariant Hamiltonian $\mathcal{H}_{\text{gauge}} = \mathcal{H}_{\text{free}} + \mathcal{H}'$ in the weak-field limit, defining $\theta_{ij} \approx aA_\mu(x_i)$. The first-order dynamic perturbation is given by:

$$\mathcal{H}' = -ia \sum_{\langle i,j \rangle} A_\mu \left(c_i^\dagger c_j - c_j^\dagger c_i \right). \tag{14}$$

Because the internal C_8 hopping links intentionally carry zero gauge phase under this simplified linear projection, the perturbation $\mathcal{H}'$ maintains support strictly and exclusively on the external square-bridge links governed by $\mathcal{H}_{\text{ext}}$.

Projecting $\mathcal{H}'$ between the unperturbed Γ-point eigenstates $\langle A_1|$ and $|E_x\rangle$ is algebraically identical to evaluating the sum in Appendix A, except that the Taylor-expanded static momentum phase k_x is replaced directly by the continuous dynamic vector field parameter aA_x. Therefore, up to a global lattice constant scaling, we find:

$$\langle A_1|\mathcal{H}'|E_x\rangle = -\frac{i}{2}aA_x. \tag{15}$$

Combining orthogonal spatial components exactly recovers the fundamental interaction vertex $\mathcal{H}_{\text{int}} \propto i(\mathbf{A}\cdot\mathbf{E})$. This rigorously proves that the static geometric momentum vertex and the dynamic weak-field gauge interaction correspond to identically equivalent physical inter-band matrix elements, fulfilling a structural lattice equivalent of a Ward identity.

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