Power sums of normalised beta weights and exact arcsine averages of hypercubic lattice invariants

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Abstract

Power sums of normalised positive weights measure concentration in random partitions and include the inverse participation ratio. Let $w_1, \ldots, w_d$ be independent Beta(a, a) variables, $p_\mu = w_\mu / \sum_\nu w_\nu$, and $R_n = \sum_\mu p_\mu^n$. We combine a Mellin representation of the random denominator with the beta Laplace transform to reduce the expectation $\mathbb{E}[R_n]$ to an explicit one-dimensional integral. The formula holds for every $a > 0$, $d \geq 1$ and integer $n \geq 2$. The hypergeometric differential equation gives a second expression for $\mathbb{E}[R_2]$, while an expansion about the mean total weight gives the first finite-d correction. When $a = 1/2$, the weights have the arcsine law and a direct lattice interpretation: $\mathbb{E}[R_n]$ is the flat Brillouin-zone average of $\widehat{k}^{[2n]}/(\widehat{k}^{[2]})^n$. For $d = 2$, the averages at $n = 2$ and $n = 3$ are exactly $1 - 1/\pi$ and $1 - 3/(2\pi)$. Numerical values for $d = 2, \ldots, 6$ provide reproducible benchmarks. The results connect normalised random weights with hypercubic lattice geometry. They concern flat, dimensionless averages rather than propagator-weighted Watson integrals or fixed-shell hypercubic extrapolations.

Keywords: normalised random weights; beta distribution; inverse participation ratio; Brillouin-zone average; hypercubic lattice; Watson integral

1 Introduction

Let $p_1, \ldots, p_d$ be non-negative numbers summing to one. The power sums

$$R_n = \sum_{\mu=1}^{d} p_\mu^n, \qquad n \geq 2, \tag{1}$$

quantify the degree of concentration of the vector. In particular R_2 is the inverse participation ratio (IPR). Participation ratios appear routinely in studies of localisation and multifractality [1, 2], while power sums of random weights also arise in the analysis of fragmented and multivalley systems [3]. They obey the elementary bounds $d^{1-n} \leq R_n \leq 1$, the lower extreme corresponding to complete equipartition and the upper extreme to a single non-zero component.

The same quantities admit a direct interpretation in lattice field theory. With the standard nearest-neighbour lattice momentum

$$\widehat{k}_\mu = 2\sin(k_\mu/2), \qquad k_\mu \in [-\pi, \pi], \tag{2}$$

one defines

$$R_n(k) = \frac{\widehat{k}^{[2n]}}{(\widehat{k}^{[2]})^n}, \qquad \widehat{k}^{[2n]} = \sum_{\mu=1}^{d} \widehat{k}_\mu^{2n}. \tag{3}$$

Ratios such as $\widehat{k}^{[4]}/(\widehat{k}^{[2]})^2$ distinguish distinct hypercubic orbits at fixed squared momentum and are employed to organise rotational-symmetry-breaking lattice artefacts [4, 5]. Their averages over the full Brillouin zone therefore constitute natural reference values.

When the components $k_1, \ldots, k_d$ are independent and uniform on the Brillouin zone,

$$w_\mu = \sin^2(k_\mu/2) \sim \mathrm{Beta}\left(\tfrac{1}{2}, \tfrac{1}{2}\right), \qquad p_\mu = \frac{w_\mu}{W}, \quad W = \sum_{\nu=1}^{d} w_\nu. \tag{4}$$

Consequently the Brillouin-zone average of (3) coincides with $\mathbb{E}R_n$ for independent arcsine weights. The resulting probability vector is *not* Dirichlet. Normalising independent gamma variables produces a Dirichlet vector that is independent of the total sum. This sum–proportion independence characterises the gamma family [13, 14]. Arcsine variables are bounded, and their normalised proportions remain coupled to W.

The Mellin–Laplace identity underlying the reduction is classical. Here it is developed explicitly for the symmetric-beta family and then specialised to arcsine weights and lattice invariants. Further results include integrated expressions for the quadratic invariant, exact two-dimensional values and finite-d asymptotics. These results also quantify the error in the naive equipartition estimate obtained by replacing the random denominator W with its mean.

2 Methods and exact derivations

2.1 Brillouin-zone average as an arcsine expectation

For an integrable function F, the flat zone average is

$$\langle F \rangle_{\mathrm{BZ}} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} F(k)\, \mathrm{d}^d k. \tag{5}$$

The change of variables $w = \sin^2(k/2)$ converts the uniform measure on each component into the arcsine density

$$\rho(w) = \frac{1}{\pi\sqrt{w(1-w)}}, \qquad 0 < w < 1. \tag{6}$$

The constant factor of four that appears in $\widehat{k}_\mu^2 = 4w_\mu$ cancels from the ratio (3). Hence

$$\left\langle \frac{\widehat{k}^{[2n]}}{(\widehat{k}^{[2]})^n} \right\rangle_{\mathrm{BZ}} = \mathbb{E}\left[\sum_{\mu=1}^{d} \left(\frac{w_\mu}{W}\right)^n\right] = \mathbb{E}R_n. \tag{7}$$

Although the probabilistic language is convenient for the derivation, the left-hand side emphasises that the quantity is an exact lattice-momentum average, not a model of a localised wavefunction.

2.2 Comparison with related models

Several better-known constructions also generate random probability vectors. The following comparisons distinguish them from the bounded-weight ensemble studied here.

Gamma-normalised Dirichlet weights. Independent gamma variables sharing a common scale and shape parameter a produce, after normalisation, a symmetric Dirichlet vector that is independent of the total sum. The corresponding power-sum expectations are elementary:

$$\mathbb{E}_{\mathrm{Dir}(a)} R_n = d\frac{(a)_n}{(da)_n}. \tag{8}$$

Independence of the proportions and the total is special to the gamma family [13, 14]. In particular, it does not follow merely from beta-type endpoint singularities. Transforming the arcsine vector $(w_1, \dots, w_d)$ to $(W, p_1, \dots, p_{d-1})$ yields the density

$$g_d(p) = \frac{1}{\pi^d} \prod_{\mu=1}^{d} p_\mu^{-1/2} \int_0^{1/p_{\max}} s^{d/2-1} \prod_{\mu=1}^{d} (1 - s p_\mu)^{-1/2} \, \mathrm{d}s, \qquad p_{\max} = \max_\mu p_\mu, \tag{9}$$

on the simplex. The integral factor depends on the entire composition p and on its largest component, so g_d is not the Dirichlet density with parameters $(\frac{1}{2}, \dots, \frac{1}{2})$ even though the prefactor $\prod_\mu p_\mu^{-1/2}$ is the same.

Poisson–Dirichlet and fragmentation weights. Poisson–Dirichlet laws describe infinite ranked sequences of random weights and arise as limits of finite random partitions or as normalised jumps of subordinators [15, 16]. For $P \sim \mathrm{PD}(\alpha, \theta)$ one has the standard identity

$$\mathbb{E} \sum_{j \geq 1} P_j^n = \frac{\Gamma(\theta + 1)\Gamma(n - \alpha)}{\Gamma(\theta + n)\Gamma(1 - \alpha)}, \qquad n > \alpha. \tag{10}$$

Sequential fragmentation and multivalley models likewise generate dependent weights and can produce singular participation-ratio distributions [3]. The ensemble considered here is different: it comprises exactly d exchangeable components obtained by normalising bounded independent variables. Its fixed-n power sums decay as d^{1-n}, with an explicit finite-d correction derived in Section 2.6, rather than approaching the non-degenerate infinite-partition statistics of (10).

Hypercubic invariant analyses. Existing treatments of the hypercubic group $H(d)$ employ invariants such as $\widehat{k}^{[4]}$ and $\widehat{k}^{[6]}$ to label lattice orbits and to fit or extrapolate rotational-symmetry-breaking artefacts [4, 5]. Their aim is to recover continuum-like observables at a fixed momentum scale. This programme belongs to the wider theory of lattice improvement initiated by Symanzik and developed for gauge actions by Lüscher and Weisz [6, 7]. Capitani reviews the standard perturbative techniques [8]. Our calculation is complementary: it averages over the entire Brillouin zone and supplies exact reference values for the dimensionless orbit-shape ratios (3). It neither conditions on a fixed-momentum shell nor proposes a substitute for action improvement or hypercubic extrapolation.

Watson integrals and lattice Green functions. The appearance of a full Brillouin-zone integral also invites comparison with the classical Watson integrals and their extensions [9, 10, 11, 12]. Those quantities are lattice Green functions: in the massless hypercubic case their integrands contain a propagator denominator proportional to $[d - \sum_\mu \cos(k_\mu)]^{-1}$, and the integral has the familiar convergence threshold $d > 2$. By contrast, (7) uses the flat measure to average a bounded, dimensionless ratio of homogeneous momentum invariants. It is finite for every $d \geq 1$; the point $k = 0$, where the ratio is undefined, has measure zero. Thus the two classes of integrals concern the same lattice geometry but answer different questions: propagation in the Watson case and the distribution of momentum direction among coordinate axes here.

Laplace reduction for normalised positive variables. The introduction of an auxiliary integral representation for a negative power of the total is a standard Mellin–Laplace device. Closely related Laplace-functional methods are fundamental in the theory of normalised random measures with independent increments [17]. Theorem 1 is therefore not claimed as a new transform identity. This work instead develops the transform explicitly for the symmetric-beta family and its arcsine/Bessel specialisation. It also derives the integrated quadratic identities, the closed two-dimensional values and the correction in (36). Finally, it identifies the arcsine case with a full-zone hypercubic average.

2.3 One-dimensional reduction

Write

$$f(t) = \mathbb{E}(e^{-tw}), \qquad m_n(t) = \mathbb{E}(w^n e^{-tw}) = (-1)^n f^{(n)}(t). \tag{11}$$

For the arcsine density (6) Euler's integral representation of the confluent hypergeometric function supplies

$$f(t) = e^{-t/2} I_0(t/2) = {}_1F_1\left(\tfrac{1}{2}; 1; -t\right), \tag{12}$$

$$m_n(t) = \frac{(\frac{1}{2})_n}{n!}\, {}_1F_1\left(n + \tfrac{1}{2}; n+1; -t\right). \tag{13}$$

Here $(a)_n$ denotes the rising factorial and I_ν the modified Bessel function of the first kind. The Kummer–Bessel identity in (12) and the asymptotic formulae employed below may be found in [18, sections 13.6 and 10.40].

Theorem 1 (one-dimensional representation)**.** *For integers $d \geq 1$ and $n \geq 2$,*

$$\boxed{\mathbb{E}R_n = \frac{d}{\Gamma(n)} \int_0^\infty t^{n-1} m_n(t) f(t)^{d-1}\, \mathrm{d}t.} \tag{14}$$

The integral converges absolutely. The explicit expressions (12)–(13) allow the integrand to be evaluated without numerical differentiation.

Proof. By permutation symmetry $\mathbb{E}R_n = d\mathbb{E}(w_1^n/W^n)$. Since $W > 0$ almost surely,

$$W^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} e^{-tW} \, \mathrm{d}t. \tag{15}$$

Non-negativity of the integrand permits interchange of expectation and integration by Tonelli's theorem. Independence then factors the expectation:

$$\mathbb{E}(w_1^n e^{-tW}) = \mathbb{E}(w_1^n e^{-tw_1}) \prod_{\mu=2}^{d} \mathbb{E}(e^{-tw_\mu}) = m_n(t) f(t)^{d-1}, \tag{16}$$

which establishes (14).

To confirm convergence at infinity, use the inverse-square-root singularity of the arcsine density at the origin. It implies

$$f(t) \sim (\pi t)^{-1/2}, \qquad m_n(t) \sim \frac{\Gamma(n+\frac{1}{2})}{\pi} t^{-n-1/2}. \tag{17}$$

The integrand is therefore $O(t^{-(d+2)/2})$ and is bounded near $t = 0$, so the integral converges for every $d \geq 1$. $\square$

2.4 Extension to symmetric beta weights

The reduction is not confined to the arcsine endpoint singularity. Let $w_1, \ldots, w_d$ be independent with the symmetric beta density

$$\rho_a(w) = \frac{w^{a-1}(1-w)^{a-1}}{B(a,a)}, \qquad 0 < w < 1, \quad a > 0, \tag{18}$$

and retain $p_\mu = w_\mu/W$ and $R_n = \sum_\mu p_\mu^n$.

Proposition 1 (symmetric-beta representation)**.** *For $a > 0$, $d \geq 1$ and integers $n \geq 2$,*

$$f_a(t) = \mathbb{E}(e^{-tw}) = {}_1F_1(a; 2a; -t), \tag{19}$$

$$m_{a,n}(t) = \mathbb{E}(w^n e^{-tw}) = \frac{(a)_n}{(2a)_n} {}_1F_1(a+n; 2a+n; -t), \tag{20}$$

$$\mathbb{E}_a R_n = \frac{d}{\Gamma(n)} \int_0^\infty t^{n-1} m_{a,n}(t) f_a(t)^{d-1} \, \mathrm{d}t. \tag{21}$$

The integral is absolutely convergent. The arcsine formulae (12)–(14) are recovered at $a = 1/2$.

Proof. Euler's beta integral gives (19) and (20); the Mellin–Laplace argument in Theorem 1 then gives (21). As $t \to \infty$,

$$f_a(t) \sim \frac{\Gamma(2a)}{\Gamma(a)} t^{-a}, \qquad m_{a,n}(t) \sim \frac{\Gamma(2a)\Gamma(a+n)}{\Gamma(a)^2} t^{-a-n}. \tag{22}$$

The integrand in (21) is therefore $O(t^{-ad-1})$, which is integrable for all $a, d > 0$. $\square$

The beta transform obeys

$$t f_a''(t) + (t + 2a) f_a'(t) + a f_a(t) = 0. \tag{23}$$

This differential equation supplies a compact companion formula for the quadratic power sum.

Proposition 2 (quadratic beta identity)**.** *If $ad > 1$, then*

$$\boxed{\mathbb{E}_a R_2 = 2a - (ad - 1) \int_0^\infty f_a(t)^d \, \mathrm{d}t.} \tag{24}$$

Proof. Multiply (23) by f_a^{d-1} and integrate. The condition $ad > 1$ and (22) imply $[t f_a(t)^d]_0^\infty = 0$. Hence

$$\int_0^\infty t f_a' f_a^{d-1} \, \mathrm{d}t = -\frac{1}{d} \int_0^\infty f_a^d \, \mathrm{d}t, \qquad \int_0^\infty f_a' f_a^{d-1} \, \mathrm{d}t = -\frac{1}{d}. \tag{25}$$

Finally, (21) at $n = 2$ identifies $d \int_0^\infty t f_a'' f_a^{d-1} \, \mathrm{d}t$ with $\mathbb{E}_a R_2$, giving (24). □

For the arcsine law, (24) becomes, for $d > 2$,

$$\mathbb{E} R_2 = 1 - \frac{d-2}{2} \int_0^\infty f(t)^d \, \mathrm{d}t. \tag{26}$$

Equation (24) does not, however, close to a scalar recursion relating $\mathbb{E}_a R_2$ at consecutive dimensions: the integral $\int f_a^d$ remains dimension dependent.

Returning to the arcsine case, the modified-Bessel recurrence relations express m_n for integer n as a finite linear combination of $e^{-t/2} I_0(t/2)$ and $e^{-t/2} I_1(t/2)$ with rational coefficients in t. The hypergeometric form (13) is ordinarily preferable for numerical work because it avoids cancellation associated with repeated differentiation.

2.5 The quadratic invariant

When $n = 2$ the reduction admits an alternative expression involving only the first derivative of f.

Corollary 1 (IPR representation)**.** *For $d \geq 1$,*

$$\boxed{\mathbb{E} R_2 = 1 - d(d-1) \int_0^\infty t f(t)^{d-2} f'(t)^2 \, \mathrm{d}t,} \tag{27}$$

where

$$f'(t) = \frac{1}{2} e^{-t/2} \left[I_1(t/2) - I_0(t/2) \right]. \tag{28}$$

Equivalently,

$$\mathbb{E} \sum_{\mu=1}^{d} \left(p_\mu - \frac{1}{d} \right)^2 = \mathbb{E} R_2 - \frac{1}{d}. \tag{29}$$

Proof. At $n = 2$ the integral in (14) contains $\int_0^\infty t f'' f^{d-1}\,\mathrm{d}t$. Integration by parts yields

$$\int_0^\infty t f'' f^{d-1}\,\mathrm{d}t = \left[t f' f^{d-1}\right]_0^\infty - \int_0^\infty f' f^{d-1}\,\mathrm{d}t - (d-1)\int_0^\infty t f^{d-2} f'^2\,\mathrm{d}t \tag{30}$$

$$= \frac{1}{d} - (d-1)\int_0^\infty t f^{d-2} f'^2\,\mathrm{d}t. \tag{31}$$

The boundary term vanishes by the asymptotics (17), while $\int_0^\infty f' f^{d-1}\,\mathrm{d}t = [f^d/d]_0^\infty = -1/d$. Multiplication by d produces (27); (29) is an algebraic identity. ☐

The prefactor $d(d-1)$ immediately settles the case $d = 1$: $p_1 = 1$ and $R_n = 1$. For $d > 1$, the integral is positive, consistent with $\mathbb{E}R_2 < 1$. The lower bound $\mathbb{E}R_2 \geq 1/d$ follows from the Cauchy–Schwarz inequality and does not require a delicate integral estimate.

2.5.1 Closed values in two dimensions

The quadratic formula can be evaluated in closed form when $d = 2$.

Proposition 3. *For two independent arcsine weights,*

$$\mathbb{E}R_2 = 1 - \frac{1}{\pi}, \qquad \mathbb{E}R_3 = 1 - \frac{3}{2\pi}. \tag{32}$$

Proof. Let a and b be independent copies of the arcsine law and define

$$G(a) = \mathbb{E}_b \frac{1}{a+b} = \frac{1}{\sqrt{a(a+1)}}. \tag{33}$$

The second equality follows by writing $b = \sin^2\theta$ and evaluating the elementary integral over $[0, \pi/2]$. Differentiation then gives

$$\mathbb{E}_b \frac{b}{(a+b)^2} = G(a) + aG'(a) = \frac{1}{2\sqrt{a}(1+a)^{3/2}}. \tag{34}$$

Consequently

$$\mathbb{E}\frac{ab}{(a+b)^2} = \frac{1}{2\pi}\int_0^1 \frac{\mathrm{d}a}{\sqrt{1-a}(1+a)^{3/2}} = \frac{1}{2\pi}. \tag{35}$$

(The substitution $u = \sqrt{(1-a)/(1+a)}$ converts the integral to unity.) With $p = a/(a+b)$ one has $R_2 = p^2 + (1-p)^2 = 1 - 2p(1-p)$, which yields the first identity. The elementary relation $p^3 + (1-p)^3 = 1 - 3p(1-p)$ yields the second. ☐

2.6 Large-dimension behaviour

Replacing W by its mean $d\mathbb{E}w = d/2$ recovers the leading large-d behaviour, but the first correction requires retention of the correlations between w_1^n and W. Symmetric beta variables are bounded, so a Taylor expansion about the mean can be controlled by standard concentration arguments throughout the family.

Proposition 4 (symmetric-beta finite-d correction)**.** *For fixed $a > 0$ and integer $n \geq 2$,*

$$\boxed{\mathbb{E}_a R_n = \frac{2^n (a)_n}{(2a)_n}\, d^{1-n} \left[1 + \frac{C_{a,n}}{d} + O(d^{-2})\right].} \qquad C_{a,n} = \frac{n(n+1)}{2(2a+1)} - \frac{n^2}{2a+n}. \tag{36}$$

For the arcsine case $a = 1/2$, this reduces to

$$\mathbb{E}R_n = \frac{2^n (\frac{1}{2})_n}{n!}\, d^{1-n} \left[1 + \frac{n(n-1)^2}{4(n+1)d} + O(d^{-2})\right]. \tag{37}$$

In particular,

$$\mathbb{E}R_2 = \frac{3}{2d} + \frac{1}{4d^2} + O(d^{-3}). \tag{38}$$

Proof. Write $\mu = \mathbb{E}w$, $\sigma^2 = \mathrm{var}(w)$, $b_r = \mathbb{E}w^r$ and

$$\Delta_d = \frac{W - d\mu}{d\mu}. \tag{39}$$

Then

$$\mathbb{E}R_n = d^{1-n}\mu^{-n}\mathbb{E}\left[w_1^n (1+\Delta_d)^{-n}\right]. \tag{40}$$

On the event $|\Delta_d| \leq 1/2$, expand the final factor in (40) through third order, with a fourth-order remainder bounded by a constant times $|\Delta_d|^4$. Boundedness and independence of the beta variables give

$$\mathbb{E}(w_1^n \Delta_d^3) = O(d^{-2}), \qquad \mathbb{E}(w_1^n |\Delta_d|^4) = O(d^{-2}). \tag{41}$$

Thus only the linear and quadratic terms contribute at relative order d^{-1}. The complementary event has exponentially small probability by Hoeffding's inequality. More explicitly, if $A_d = \{|\Delta_d| \leq 1/2\}$, then $\mathbb{E}[R_n \mathbf{1}_{A_d^c}] \leq \Pr(A_d^c)$ because $R_n \leq 1$; hence its contribution is exponentially small as well. The two required moments are

$$\mathbb{E}(w_1^n \Delta_d) = \frac{b_{n+1} - \mu b_n}{d\mu}, \tag{42}$$

$$\mathbb{E}(w_1^n \Delta_d^2) = \frac{\sigma^2 b_n}{d\mu^2} + O(d^{-2}). \tag{43}$$

Hence

$$\mathbb{E}\left[w_1^n (1+\Delta_d)^{-n}\right] = b_n + \frac{1}{d}\left[-\frac{n(b_{n+1} - \mu b_n)}{\mu} + \frac{n(n+1)\sigma^2 b_n}{2\mu^2}\right] + O(d^{-2}). \tag{44}$$

For the symmetric beta law one has

$$\mu = \frac{1}{2}, \qquad \sigma^2 = \frac{1}{4(2a+1)}, \qquad b_n = \frac{(a)_n}{(2a)_n}, \qquad \frac{b_{n+1}}{b_n} = \frac{a+n}{2a+n}. \tag{45}$$

Substitution into (44) gives $C_{a,n}$ and establishes (36). Setting $a = 1/2$ yields (37). $\square$

The leading coefficient in (37) exceeds the pure equipartition value d^{1-n} because the un-normalised arcsine components remain broad. For $n = 2$ one finds $d\mathbb{E}R_2 \to 3/2$, whereas equipartition would give unity. The positive $1/(4d)$ correction to $d\mathbb{E}R_2$ is already visible at moderate dimension.

3 Results: numerical benchmarks

Table 1 was obtained from the integral representation (14) together with the closed transforms (13). The integration range was split and the far tail mapped by $t = 1/u$ to avoid nested quadrature across the endpoint singularities of the original beta density. The $d = 2$ entries for $n = 2, 3$ reproduce Proposition 3; the $n = 2$ column was also recomputed independently from (27). The digits displayed are rounded, not fitted. Quadrature was performed with 80-decimal-digit working precision; changing the interval decomposition and tail-matching point altered every entry by less than 10^{-40}, well below the 15 digits reported. The accompanying verifier also compares the $d = 4$, $n = 3, 4$ entries with one million seeded direct arcsine samples. It tests the finite-d coefficient in Proposition 4 for $n = 2, 3, 4$ and checks the symmetric-beta reduction at representative non-arcsine parameter values. The companion quadratic identity is checked at the same time.

d	$\mathbb{E}R_2$	$\mathbb{E}R_3$	$\mathbb{E}R_4$
2	0.681690113816209	0.522535170724314	0.429694787254042
3	0.494537980282674	0.290463604756102	0.192481136520490
4	0.380266439075759	0.174672457023530	0.092882333806077
5	0.306215125095861	0.113186654996544	0.048740232104541
6	0.255357751182219	0.078158144689741	0.027736241018425

Table 1: Flat Brillouin-zone averages $\mathbb{E}R_n = \left\langle \widehat{k}^{[2n]}/(\widehat{k}^{[2]})^n \right\rangle_{\mathrm{BZ}}$ of the normalised hypercubic invariants defined in (3).

For the physically common case $d = 4$, the first entry states that the flat-zone average of $\widehat{k}^{[4]}/(\widehat{k}^{[2]})^2$ is approximately 0.3802664391. An average restricted to a shell of fixed $\widehat{k}^{[2]}$ is a different quantity because the components are then dependent. This distinction matters when comparing the result with orbit cuts or shell-by-shell hypercubic extrapolations. The standard lattice-perturbation and Watson-integral references cited above tabulate propagator-weighted or action-dependent Brillouin-zone constants; we have not found the dimensionless flat-measure constants in Table 1 in those tabulations. The point is to distinguish the definitions, not to claim that the table replaces those established integral families.

4 Discussion

The arcsine law appears for a purely geometric reason: it is the push-forward of a uniformly sampled lattice-momentum component under the map $k \mapsto \sin^2(k/2)$. This observation converts a d-dimensional zone average of a ratio into the expectation of a normalised random vector. The transform representation then removes the denominator while retaining its fluctuations.

Three features of the result are useful in practice. First, (21) is valid at any finite dimension and for every integer $n \geq 2$, while retaining a tunable endpoint and concentration parameter a. Second, the integrated forms (24) and (27) supply independent and numerically stable expressions for the most frequently used invariant.

Third, (36) cleanly separates the universal d^{1-n} scaling from coefficients fixed by the symmetric-beta moments.

More generally, the method applies to independent non-negative weights whose Laplace transform is known, although the physical interpretation then depends on the origin of those weights. Natural extensions include action-weighted zone averages and conditional averages on fixed lattice-momentum shells. Neither extension factorises component by component, so each requires additional structure beyond the present one-dimensional reduction.

Data availability statement

All data that support the findings of this study are included within the article. The supplementary file `verify_arcsine_note_revised.py` reproduces the analytic formulae and Table 1, and performs the independent sampling and finite-d checks described above. Its SHA-256 checksum is `98f3e40375f8e24dc2be88bcb179b8c8 5ac9ea55783d6915eb7f0ce09b14438b`.

Acknowledgments

The author declares no conflicts of interest. This research received no specific grant from any funding agency in the public, commercial or not-for-profit sectors. OpenAI Codex (GPT-5, accessed August 2026) was used during revision to assist with literature discovery, language editing and computational cross-checking. The author independently verified the citations, derivations and numerical results, approved the final text and takes full responsibility for the work.

Ethical statement

This theoretical study involved no human participants, animals, human data or tissue. Ethical approval was therefore not required.

References

[1] Y. V. Fyodorov and A. D. Mirlin, "Analytical derivation of the scaling law for the inverse participation ratio in quasi-one-dimensional disordered systems," *Phys. Rev. Lett.* **69**, 1093–1096 (1992), https://doi.org/10.1103/PhysRevLett.69.1093.

[2] F. Evers and A. D. Mirlin, "Anderson transitions," *Rev. Mod. Phys.* **80**, 1355–1417 (2008), https://doi.org/10.1103/RevModPhys.80.1355.

[3] B. Derrida and H. Flyvbjerg, "Statistical properties of randomly broken objects and of multivalley structures in disordered systems," *J. Phys. A: Math. Gen.* **20**, 5273–5288 (1987), https://doi.org/10.1088/0305-4470/20/15/039.

[4] Ph. Boucaud, F. de Soto, J. P. Leroy, A. Le Yaouanc, J. Micheli, H. Moutarde, O. Pène and J. Rodríguez-Quintero, "Quark propagator and vertex: systematic corrections of hypercubic artifacts from lattice simulations," *Phys. Lett. B* **575**, 256–267 (2003), https://doi.org/10.1016/j.physletb.2003.08.065.

[5] F. de Soto and C. Roiesnel, "On the reduction of hypercubic lattice artifacts," *J. High Energy Phys.* **09**, 007 (2007), https://doi.org/10.1088/1126-6708/2007/09/007.

[6] K. Symanzik, "Continuum limit and improved action in lattice theories. I. Principles and ϕ^4 theory," *Nucl. Phys. B* **226**, 187–204 (1983), https://doi.org/10.1016/0550-3213(83)90468-6.

[7] M. Lüscher and P. Weisz, "On-shell improved lattice gauge theories," *Commun. Math. Phys.* **97**, 59–77 (1985), Erratum **98**, 433 (1985), https://doi.org/10.1007/BF01206178.

[8] S. Capitani, "Lattice perturbation theory," *Phys. Rep.* **382**, 113–302 (2003), https://doi.org/10.1016/S0370-1573(03)00211-4.

[9] G. N. Watson, "Three triple integrals," *Q. J. Math.* **10**, 266–276 (1939), https://doi.org/10.1093/qmath/os-10.1.266.

[10] M. L. Glasser and I. J. Zucker, "Extended Watson integrals for the cubic lattices," *Proc. Natl. Acad. Sci. USA* **74**, 1800–1801 (1977), https://doi.org/10.1073/pnas.74.5.1800.

[11] G. S. Joyce and I. J. Zucker, "Evaluation of the Watson integral and associated logarithmic integral for the d-dimensional hypercubic lattice," *J. Phys. A: Math. Gen.* **34**, 7349–7354 (2001), https://doi.org/10.1088/0305-4470/34/36/314.

[12] I. J. Zucker, "70+ years of the Watson integrals," *J. Stat. Phys.* **145**, 591–612 (2011), https://doi.org/10.1007/s10955-011-0273-0.

[13] E. Lukacs, "A characterization of the gamma distribution," *Ann. Math. Stat.* **26**, 319–324 (1955), https://doi.org/10.1214/aoms/1177728549.

[14] R. J. Connor and J. E. Mosimann, "Concepts of independence for proportions with a generalization of the Dirichlet distribution," *J. Am. Stat. Assoc.* **64**, 194–206 (1969), https://doi.org/10.1080/01621459.1969.10500963.

[15] J. F. C. Kingman, "Random discrete distributions," *J. R. Stat. Soc. B* **37**, 1–15 (1975), https://doi.org/10.1111/j.2517-6161.1975.tb01024.x.

[16] J. Pitman and M. Yor, "The two-parameter Poisson–Dirichlet distribution derived from a stable subordinator," *Ann. Probab.* **25**, 855–900 (1997), https://doi.org/10.1214/aop/1024404422.

[17] E. Regazzini, A. Lijoi and I. Prünster, "Distributional results for means of normalized random measures with independent increments," *Ann. Stat.* **31**, 560–585 (2003), https://doi.org/10.1214/aos/1051027881.

[18] NIST Digital Library of Mathematical Functions, "Confluent hypergeometric functions" and "Modified Bessel functions," `https://dlmf.nist.gov/` (accessed 11 August 2026).